

# Linear imperfections.

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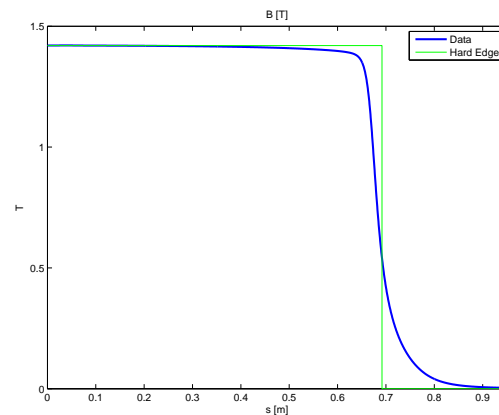
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- In previous lessons, you have learned that the particles will oscillate around the path defined by the magnetic elements.
- This path is defined by the dipolar fields (bending magnets).
- The particles are kept in place by the focusing elements (quadrupolar magnets).
- And the motion in the different planes is not coupled, and there are only linear elements.
- However, in the real world, things are not so simple.
- Let's review the problems.

# Magnet modeling

- The real magnets are not the ones used in the simulation codes. Even if the hard-edge model is a good approximation, is not exact. There are the effects of fringe field, changing fields in the body of the magnets, etc:



**Figure:** Magnetic field for one half of the bending magnet for the ALBA synchrotron light source

- It is possible to use more realistic models, and predict the new reference orbit and tunes.

# Magnet fabrication

- In fabrication of the magnets, more divergences from our model will happen:
- Finite precision in the shaping of the pole faces (in the order of  $\pm 20 \mu\text{m}$ ).
- Finite precision in the assembly of the magnets.

## Field

In general is the precision magnetic fields will be of the order of  $\Delta B/B = 10^{-3}$

- This effect will include additional magnetic fields, that can introduce orbit distortions, coupling between the planes, change of the optical functions, tune shifts, and non linear terms in the equation of motion.

# Magnet Installation

- When positioning the magnets in their position at the accelerator complex, the precision of the positioning is limited, either by the accuracy in the installation and the accuracy in the measuring.

## Position

In general is the precision of the position of the magnetic center will be of the order of  $\Delta x, y = 150 \mu\text{m}$  and  $\Delta s \sim 1\text{mm}$ , Angular  $\sim 100 \mu\text{rad}$ .

- The misalignment of the quadrupoles will introduce a distortion of the closed orbit, and the rotation will introduce coupling between the planes.

# Other Elements

- In both colliders and synchrotron light sources exist other magnetic elements than the dipoles and quadrupoles.
- Sextupoles are required to compensate the chromaticity.
- Solenoids.
- Insertion devices.
- They could affect the orbit, optical functions, tunes and coupling, but they also introduced nonlinear motion.

## Dynamic aperture

Reduction of the stable area of motion and energy acceptance of the machine.

- The magnets will be placed in noise environments.
- Vibrations and deformations of the ground will move the magnets.
- The magnetic field of different magnets can interfere between them.
- 50 Hz noise of the mains electrical line.
- Exotic effects: TGV in LEP, Tide of the moon, earthquakes, setting the energy of the machine, . . .

## The job of the accelerator physicist

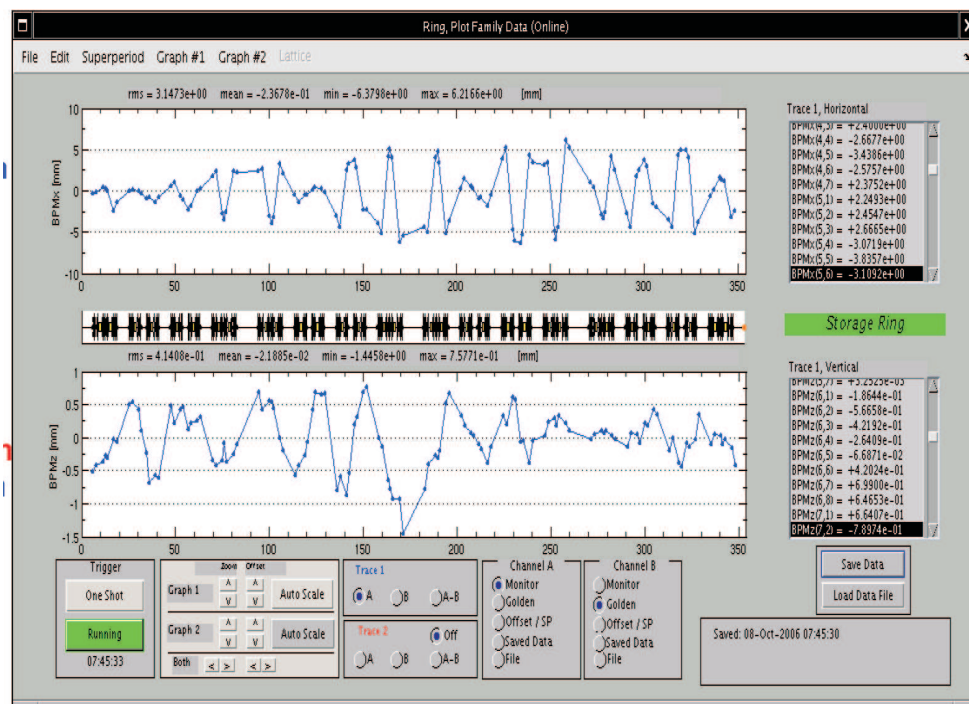
*The design of an accelerator includes the evaluation of this effects, define the tolerances allowed in the elements, and implement the correction method to compensate for them.*

*Those includes:*

- *Orbit correction system: BPMs and Corrector magnets.*
- *Measurement and correction of the tune and optical functions.*
- *Skew quadrupoles for coupling correction.*
- *Sextupoles for correcting chromaticity and for optimizing dynamic aperture.*

*In the following lectures we will offer an introduction to those jobs.*

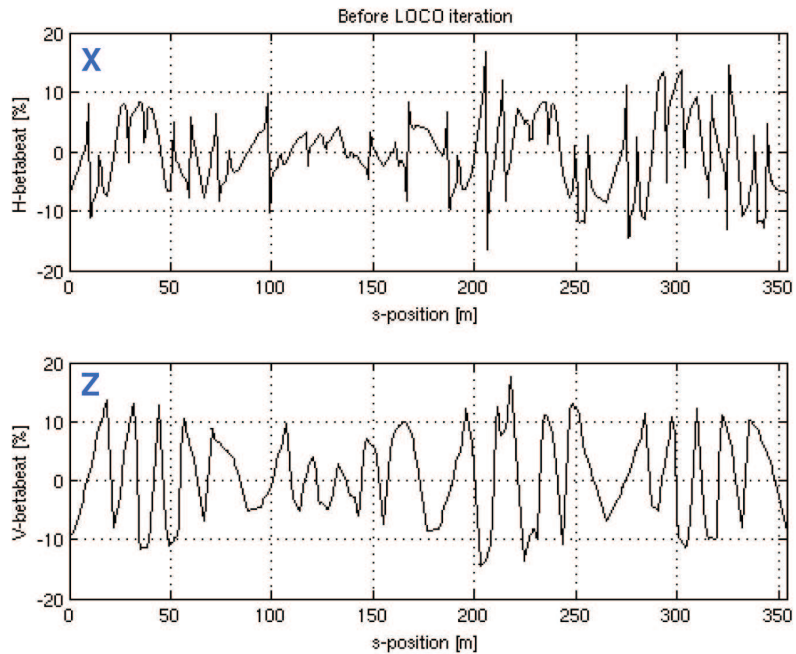
- Soleil is a 2.75 GeV, 350 m synchrotron light source. The operation of the machine has just started.
- The magnet where produced and install using state of the art technology.
- Even with all the care in the construction, a residual orbit of several mm was still present, quite close to the aperture of the vacuum chamber.
- The distortion of the optical functions and tune change is still large.
- However, they were able to cope with those errors, and now the machine is working properly, with a very stable orbit, and the proper tunes and optical functions.
- The next two slides come from real data of the machine, just in the first days of the start-up of the machine



Data courtesy from A. Nadji.



## $\beta$ -beating before correction



# Hill's equation

Some reminders of the equations of motion



# Reminder of basic equations of motion

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In previous lectures, you have learned that in the moving coordinate system, the Lorentz equation

$$\frac{d}{dt} \vec{v} = \frac{e}{m} (\vec{v} \times \vec{B}) \quad (1)$$

becomes the following two uncoupled equations:

$$\frac{d^2 x}{ds^2} - \left( k(s) - \frac{1}{\rho^2} \right) x = \frac{1}{\rho} \frac{\Delta p}{p_0} \quad (2)$$

$$\frac{d^2 y}{ds^2} + k(s) y = 0 \quad (3)$$



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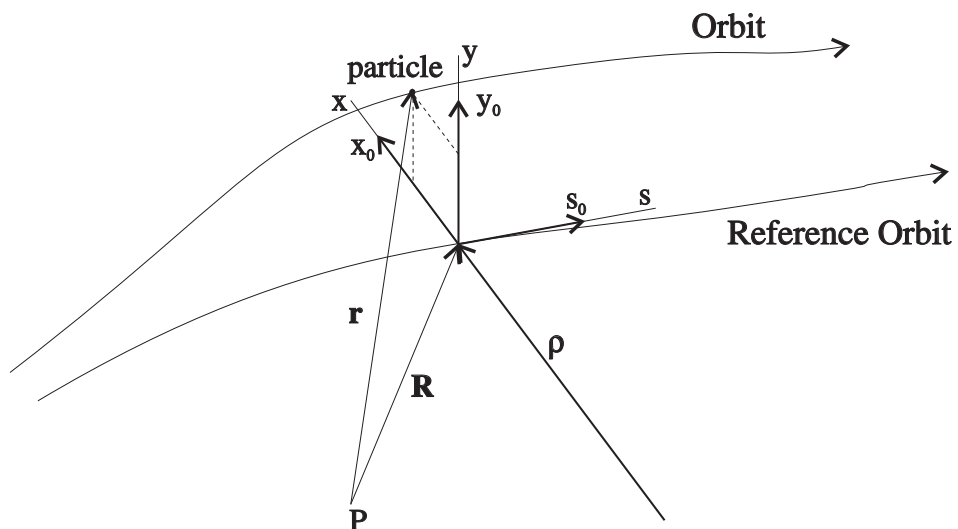


Figure: Moving coordinate system



where:

$$k(s) = \frac{1}{B\rho} \frac{\partial B_y(s)}{\partial x}$$

$\rho$  is the radius of curvature of the electrons

$\frac{\Delta p}{p_0}$  is the momentum deviation respect the reference particle

If we concentrate in the on-energy particle ( $p = p_0$ ), both equations 2 and 3, can be written as:

$$u'' + K(s)u = 0 \quad (4)$$

where  $u$  stands for  $x$  or  $y$  and  $K(s)$  is given by:

$$K(s) = \begin{cases} -\left(k(s) - \frac{1}{\rho^2}\right) & u = x \\ k(s) & u = y \end{cases} \quad (5)$$

From here we can see that is difficult to focus simultaneously in both planes



## Harmonic Oscillator

Equation 4 is the equation of an an harmonic oscillator (Hill's equation). To solve it, we can write it as:

$$\vec{u}' + \begin{pmatrix} 0 & -1 \\ K(s) & 0 \end{pmatrix} \vec{u} = 0 \quad (6)$$

where

$$\vec{u} = \begin{pmatrix} u \\ u' \end{pmatrix}$$

If  $K$  is constant, the solution can be written as:

$$\vec{u}(s) = A\vec{u}_1(s) + B\vec{u}_2(s) \quad (7)$$

with

$$\vec{u}_1(s) = \begin{pmatrix} \sin(\sqrt{K}s) \\ \sqrt{K} \cos(\sqrt{K}s) \end{pmatrix} \quad \vec{u}_2(s) = \begin{pmatrix} \cos(\sqrt{K}s) \\ -\sqrt{K} \sin(\sqrt{K}s) \end{pmatrix} \quad (8)$$



and initial conditions:

$$\vec{u}_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{u}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (9)$$

and the transport map  $M$  is given by:

$$\vec{y}(s) = M(s - s_0) \times \vec{y}(s_0) \quad (10)$$

$$M(s - s_0) = \begin{pmatrix} \cos(\sqrt{K}(s - s_0)) & \sin(\sqrt{K}(s - s_0)) \\ -\sqrt{K} \sin(\sqrt{K}(s - s_0)) & \sqrt{K} \cos(\sqrt{K}(s - s_0)) \end{pmatrix}$$

is a rotation in the phase space.

## K not constant

In case  $K$  is not constant (as in an accelerator), the Floquet theorem allows us to write the solution also as the linear combination of a “sinelike” and a “cosinelike” solutions, now with a variable amplitude and phase advance:

$$\mathbf{u}(s) = A\mathbf{u}_a(s) + B\mathbf{u}_b(s) \quad (11)$$

where

$$\mathbf{u}_a(s) = \sqrt{C} \sqrt{\beta(s)} e^{+i\phi(s)} \quad (12)$$

$$\mathbf{u}_b(s) = \sqrt{C} \sqrt{\beta(s)} e^{-i\phi(s)} \quad (13)$$

where  $C$  is a constant and  $\beta(s)$  is the  $s$ -dependent amplitude, known as optical betatron function, with units of length (usually meters). The phase term  $\phi(s)$  depends on  $\beta(s)$  as:

$$\phi(s) = \int_{s_0}^s \frac{1}{\beta(\tau)} d\tau + \phi_0 \quad (14)$$

Substituting on of the equations 12 or 13 in equation 4, the following differential equation for the beta function is obtained

$$\frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + K(s)\beta^2(s) = 1 \quad (15)$$

For a periodic system, as in the case of a storage ring  $K(s) = K(s + L)$  and the beta function is also periodic,  $\beta(s) = \beta(s + L)$ , and the total phase advance per revolution is the tune or number of oscillations:

## Definition

Tune

$$Q = \frac{1}{2\pi} \oint \frac{1}{\beta(\tau)} d\tau$$

Going back to the equations of motion, we can combine equations 12 and 13 in the two independent solutions, sinelike and cosinelike:

$$S(s) = -\frac{i}{2} (u_a(s) - u_b(s)) \quad (16)$$

$$C(s) = \frac{1}{2} (u_a(s) + u_b(s)) \quad (17)$$

with the same initial conditions.

## Definition

Twiss functions

beta function	$\beta(s)$
alpha function	$\alpha(s) \equiv -\frac{1}{2} \frac{d\beta(s)}{ds}$
gamma function	$\gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$

The sinelike and cosinelike solutions are:

$$\vec{S}(s) = \begin{pmatrix} \sqrt{\beta(s)\beta(s_0)} \sin(\phi(s) + \phi_0) \\ \frac{\sqrt{\beta(s_0)}}{\sqrt{\beta(s)}} (\cos(\phi(s) + \phi_0) + \alpha(s) \sin(\phi(s) + \phi_0)) \end{pmatrix} \quad (18)$$

$$\vec{C}(s) = \begin{pmatrix} \frac{\sqrt{\beta(s)}}{\sqrt{\beta(s_0)}} (\cos(\phi(s) + \phi_0) + \alpha(s_0) \sin(\phi(s) + \phi_0)) \\ -\frac{1 + \alpha(s)\alpha(s_0)}{\sqrt{\beta(s)\beta(s_0)}} (\sin(\phi(s) - \phi_0) + (\alpha_0 - \alpha) \cos(\phi(s) + \phi_0)) \end{pmatrix} \quad (19)$$

And now we can write the transport map  $\mathbf{M}$  from  $s_0$  to  $s$  as:

$$\vec{y}(s) = \mathbf{M}(s, s_0) \times \vec{y}(s_0) \quad (20)$$

$$\mathbf{M} = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \quad (21)$$

and for a periodic system:

$$\mathbf{M} = \mathbf{I} \cos(2\pi Q) + \mathbf{J} \sin(2\pi Q) \quad (22)$$

where  $\mathbf{I}$  is the identity matrix and  $\mathbf{J}$  is

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \quad (23)$$

Combining the equation of motion and the definition of the twiss parameters, we can obtain the following equation for the invariant of the motion:

$$C = \gamma u^2 + 2\alpha u u' + \beta u'^2 \quad (24)$$

that is the equation of an ellipse in the phase space, where the area is:

$$\text{Area} = \pi C = \pi \varepsilon \quad (25)$$

## Definition

$\varepsilon$  is the emittance

At any given point, the rms beam size and divergence can be written as:

$$\sigma(s) = \sqrt{\varepsilon \beta(s)} \quad (26)$$

$$\sigma'(s) = \sqrt{\varepsilon \gamma(s)} \quad (27)$$

## Effect of dipolar errors

The dipolar errors affects basically the orbit of the particles. Here we review the effect and how to compensate for them.

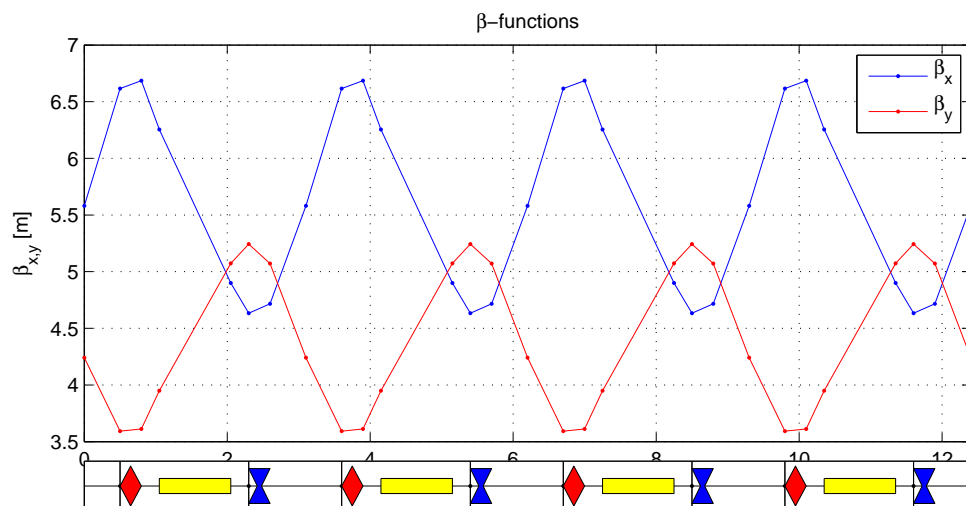
## Dipolar Errors

The most basic errors are the dipolar ones. The effect of this errors is to modify the trajectory of the particles. In absence of non linear elements, they will not affect the optical functions or the tune of the machine.

They can be caused by imperfections in the dipole magnets (either due to the magnet or the power supply or others), displacement of quadrupolar magnets, or by extra dipolar effects generated by the environmentnt.

## A example lattice

In order to have a basic understanding of the effect of the dipolar errors, we will use a FODO lattice and introduce a single dipolar kick on it. The basic cell of the lattice is composeded of two quadrupoles and a bending magnet, and the lattice has 16 cells of those.



# Single think kick example

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A single kick (a point-like change in the momentum of the particle) has been placed in the lattice. Figure shows the trajectory that a particle with initial coordinate  $(0, 0, 0, 0)$  will follow, in the real space and in the normalized phase space. In this last figure we can see the effect of the kick, and that the trajectory does not close over itself.

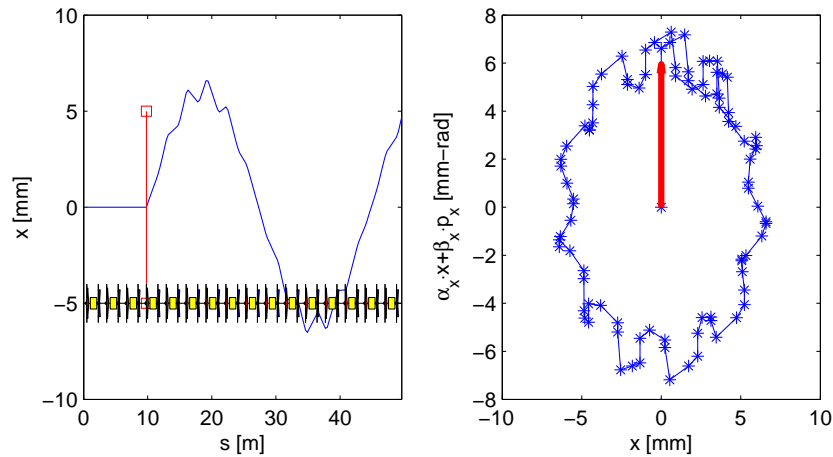


Figure: Trajectory after a kick. The red line is the kick



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The next figure shows the trajectory and the motion in the phase space for a particle which initial conditions are so that the particle returns to the same coordinates after one revolution. This is called the closed orbit.

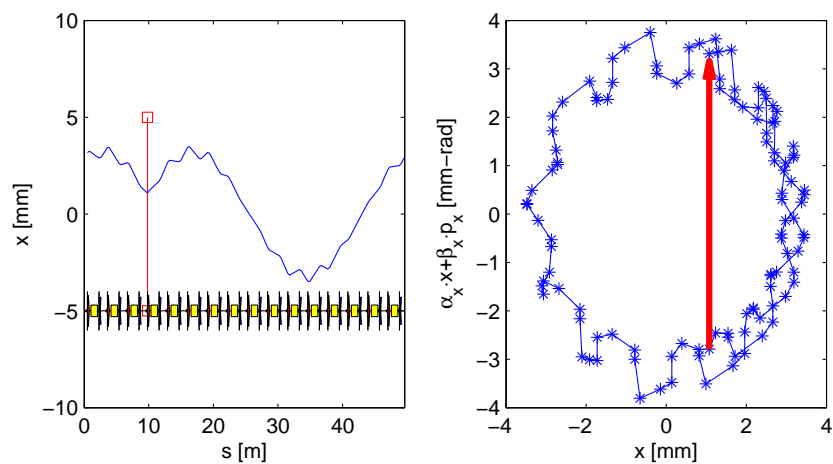


Figure: Trajectory after a kick. The red line is the kick



We recall that the trajectory of a particle moving through an accelerator can be written in term of the twiss parameters and the initial amplitude as:

$$x = \sqrt{C\beta_x} \cos \phi_x \quad (28)$$

$$p_x = \sqrt{\frac{C}{\beta_x}} (\sin \phi_x + \alpha_x \cos \phi_x) \quad (29)$$

At the position of the kick, the effect of it is to change the momentum and the trajectory after the kick will be:

$$x_{\text{after}} = x_{\text{before}} \quad (30)$$

$$p_{x,\text{after}} = p_{x,\text{before}} + \Delta\theta \quad (31)$$

Assuming that the total phase advance in the lattice after one revolution is  $\mu_x$ , then the conditions for the trajectory to be closed after one turn is:

### Closed orbit condition

$$\sqrt{C_0\beta_{x0}} \cos(\phi_{x0} + \mu_x) = \sqrt{C_0\beta_{x0}} \cos(\phi_{x0}) \quad (32)$$

$$\begin{aligned} \sqrt{\frac{C_0}{\beta_{x0}}} (\sin(\phi_{x0} + \mu_x) + \alpha_x \cos(\phi_{x0} + \mu_x)) + \Delta\theta = \\ = \sqrt{\frac{C_0}{\beta_{x0}}} (\sin \phi_{x0} + \alpha_x \cos \phi_{x0}) \quad (33) \end{aligned}$$

If we solve this system of equations for the two variables defining the initial conditions ( $C_0$  and  $\phi_{x0}$ ), we found that the solution are:

## Closed orbit solution

$$C_0 = \frac{\beta_{x0} \Delta\theta^2}{4 \sin^2 \pi Q_x}, \quad \phi_{x0} = \pi Q_x \quad (34)$$

## Resonance

The motion is not stable if the tune is an integer, and the errors are amplified if we are close to an integer.

## Dependence on $\beta$

The closed orbit depends on the value of the  $\beta$  function at the location of the error.

Using equation 34, we can write the closed orbit along the machine resulting from a single dipolar error at  $s = 0$  as:

$$x_{co}(s) = \frac{\sqrt{\beta_x(0)\beta_x(s)}}{2 \sin \pi Q_x} \Delta\theta \cos(\mu(s) + \pi Q_x) \quad (35)$$

where  $\mu(s)$  is the phase advance from  $s = 0$  to  $s$ .

# Sources of errors

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There are several possible sources of extra dipolar fields in a storage ring. The main ones are:

Table: Sources of errors

Element type	Source	$\theta$	Plane
Quadrupole (L, K)	Displacement by $\Delta u$	$K \cdot L \cdot \Delta u$	x, y
Dipole ( $\psi$ )	Rotation around s $\phi_s$	$\psi \cdot \phi_s$	y
Dipole ( $\psi$ )	Field Error $\Delta B/B$	$\psi \cdot \Delta B/B$	x



# Quadrupolar displacement

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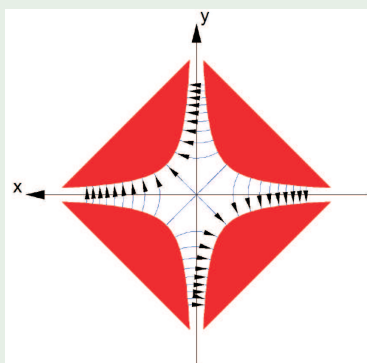
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## Quad displacement

The next figure shows the magnetic field inside a quadropole:



And the fields are:

$$B_x = b_2 \frac{y}{r_0}$$

$$B_y = b_2 \frac{x}{r_0}$$

$$B_z = 0$$

If the body of the quadrupole is displaced vertically by an amount  $\Delta y$ , then  $B_x$  component *with respect* to the reference trajectory will be:

$$B_x^{disp} = b_2 \frac{y}{r_0} - b_2 \frac{\Delta y}{r_0} \tag{36}$$

- The second term of equation 36 is independent of the coordinates and represents a dipolar field (a **kick**) superposed on the quad one, of value:

$$\Delta B_x = -b_2 \frac{\Delta y}{r_0} \quad (37)$$

- The kick is a horizontal field, so it deflects the particle in the vertical plane.
- In this case, the kick that the particle will see is (assuming  $l$  the length of the quad):

$$\Delta p_y = \Delta \theta_y = \frac{q}{p_0} \Delta B_x l \quad (38)$$

## Generalization to many kicks

- In presence of more than a single kick, as the motion is linear, we can just add the effect of the various kicks, For example, for the presence of a horizontal dipolar error  $B_x(s)$  distributed along the circumference of the machine, the vertical close orbit distortion will be:

$$u_{co}(s) = \oint \frac{\sqrt{\beta(\tau)\beta(s)}}{2 \sin \pi Q} \Delta \theta(\tau) \cos(\mu(\tau \rightarrow s) + \pi Q) d\tau \quad (39)$$

where  $\Delta \theta(\tau) = \frac{q}{p_0} B(\tau)$  is the kick generated by the error.

- But when designing an accelerator the exact distribution of errors is not know. In general we will know a possible statistical prediction of the distribution of errors around the machine:

### ALBA

For example, for the ALBA synchrotron light source, we expect the quadrupoles to be aligned with a precision of  $\pm 300 \mu\text{m}$ , or the magnetic field of the bendings to be accurate up to a relative  $5 \times 10^{-3}$ .

- When designing the accelerator, it is required to predict what will be the effect of the dipolar errors in the closed orbit.
- Is possible to one of the multiple simulation codes, but using some simplified assumptions we can provide an analytical estimate:
  - Point-like errors  $\theta_k$ .
  - Errors know through its standard deviation.  $\sigma(\theta_k)$
  - Uncorrelated errors. with zero average value.
  - N errors
- In that case, it can be show that the standart deviation of the close orbit will be given by:

$$\sigma(u(s)) = \langle u^2(s) \rangle = \frac{\sqrt{\beta(s)}}{2\sqrt{2}\sin\pi Q} \sqrt{N \langle \beta_k \rangle} \sigma(\theta_k) \quad (40)$$

From the previous equation is practical to define the amplification factor:

### Amplification Factor

Ratio of the rms closed orbit at the observation point to the rms amplitude of the source of the error:

$$A_k(s) = \frac{\sigma(u(s))}{\sigma(\text{Error})} \quad (41)$$

This magnitude is very useful when designing a machine, allowing us to set a budget between the different sources of error.

## Orbit correction

The control and correction of the closed orbit is one of the most fundamental aspects of the operation of one storage ring, both for synchrotron light sources or colliders. In a synchrotron light source, we want to provide the beam to the users accurately down the beamlines to the users, and in a collider we have to make sure that the beams collide. In modern machines, this requires an stability of the closed orbit down to the sub- $\mu\text{m}$ . Additionally, orbit control is also the first step in the correction of focusing error and coupling; and is required for a good beamlifetime

To correct the orbit, two components are needed:

- 1 Monitors to detect the position of the particles. Usually are the *beam position monitors or BPMs*, but is also possible to use the synchrotron light to measure the position of the particles. Modern BPM system offer sub- $\mu\text{m}$  resolutions in the measurement of the closed orbit, offering sampling rates from the 10 kHz (or more) to the Hz; or offer the possibility to measure the orbit turn by turn.
- 2 Actuators to compensate for the orbit deviation. These *steering or corrector magnets* allow the operator to apply dipolar field to compensate for dipolar error along the machine. Can be extra dipolar magnets, or extra coils in some of the existing magnets.
- 3 Several methods are possible, we will review some of them

If  $\Delta\vec{c}$  is a vector containing the change in the setting of one corrector magnet, and  $\Delta\vec{u}$  is the change in the reading of the the BPMs, then the response matrix  $\mathbf{M}$  is defined from:

$$\Delta\vec{u} = \mathbf{M} \times \vec{c} \quad (42)$$

$$\mathbf{M}_{i,j} = \frac{\Delta\text{BPM}(u_i)}{\Delta\text{CM}(c_j)} \quad (43)$$

$$= \frac{\sqrt{\beta_i\beta_j}}{2 \sin \pi Q} \cos(\pi Q - |\phi_i - \phi_j|) \quad (44)$$

$\mathbf{M}$  can be calculated from the model of the machine, using one of our accelerator physics code, or may be measured in the real machine, changing the setting of the correctors and recording the change in BPM readings.

- The most direct method of correction the orbit consist in inverting  $\mathbf{M}$  to predict the change in corrector magnets to compensate an measured closed orbit.
- This is not in general possible:  $\mathbf{M}$  is not square in general, or can be degenerated.
- Clever algorithms such as Singular Value Decomposition can be used to perform a pseudo-inversion of the response matrix
- Methods not based in the inversion of  $\mathbf{M}$ , such as local orbit bumps will help in those cases, or when for some reason we do not know the response matrix.

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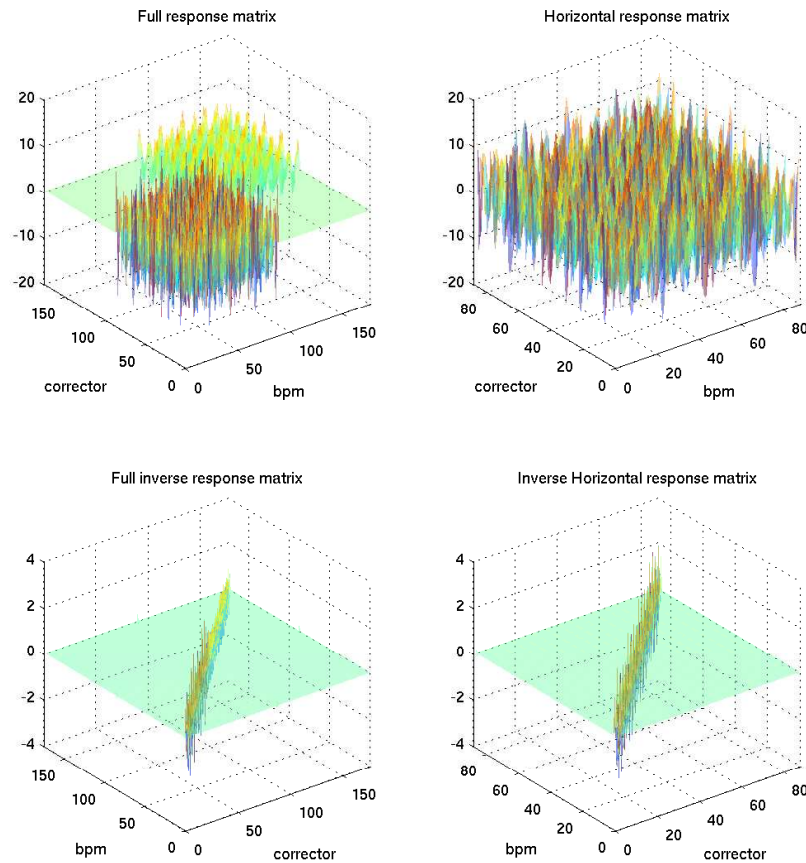


Figure: Response matrices for the ALBA machine.

# Singular Value Decomposition

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SVD is algorithm to invert non-square matrices. A matrix  $\mathbf{R}$  of dimensions  $M \times N$  can be written as:

## SVD decomposition

$$\mathbf{R} = \mathbf{U} \times \mathbf{W} \times \mathbf{V}^T \quad (45)$$

where:

- $\mathbf{U}$  is an unitary matrix ( $\mathbf{U} \times \mathbf{U}^T = \mathbf{U}^T \times \mathbf{U} = \mathbf{I}$ ) of dimensions  $M \times M$ .
- $\mathbf{V}$  is an unitary matrix ( $\mathbf{V} \times \mathbf{V}^T = \mathbf{V}^T \times \mathbf{V} = \mathbf{I}$ ) of dimensions  $N \times N$ .
- $\mathbf{W}$  is a diagonal matrix of dimensions  $M \times N$  where all the elements are positives or null, and are called the SVD eigenvalues.

Applying the SVD decomposition to the response matrix  $\mathbf{M}$  we have:

$$\Delta x = \mathbf{M} \times \Delta c \quad (46)$$

$$= \mathbf{U}_M \times \mathbf{W}_M \times \mathbf{V}_M^T \Delta c \quad (47)$$

$$\mathbf{U}_M^T \Delta x = \mathbf{W}_M \times \mathbf{V}_M^T \Delta c \quad (48)$$

and we can interpret that the SVD decomposition as transforming the vectors of the position and corrector settings to a new base where:

$$\Delta x^t = \mathbf{U}_M^T \Delta x \quad (49)$$

$$\Delta c^t = \mathbf{V}_M^T \Delta c \quad (50)$$

$$\Delta x^t = \mathbf{W}_M \Delta c^t \quad (51)$$

and in this base is easy to do a pseudo-inversion of the response matrix:



## SVD inversion

$$\Delta c = \mathbf{M}^{-1} \Delta x \quad (52)$$

$$= \mathbf{V}_M \times \mathbf{W}_M^{-1} \times \mathbf{U}_M^T \times \Delta x \quad (53)$$

where  $\mathbf{W}_M^{-1}$  is a diagonal matrix of dimensions  $N \times M$  where the elements in the diagonal are defined by:

$$W_{M,n}^{-1} = \begin{cases} 0, & W_{M,n} \leq \varepsilon W_{M,max} \\ \frac{1}{W_{M,n}}, & \text{for others} \end{cases} \quad (54)$$

where  $\varepsilon$  allows us to control the number of eigenvalues to be used.



# Some properties of the SVD method

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- Using  $\varepsilon$ , it is possible to select the numbers of eigenvalues to be used, allowing to find a balance between the strength of the correctors and the “smoothness” of the residual orbit.
- Finds the solution that minimizes the RMS value of the correctors.
- Finds the solution that minimizes the orbit at the BPMs.
- There is a lot of information about system that can be obtained, for example which correctors are the most efficient, or if they are redundant correctors or BPMs.

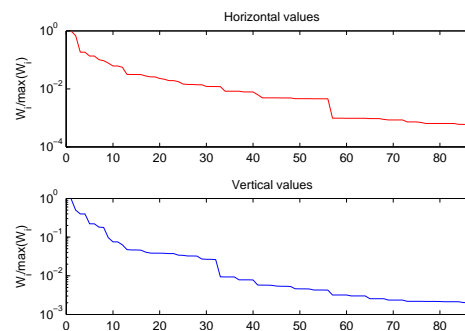


Figure: SVD eigenvalues for the ALBA response matrix.



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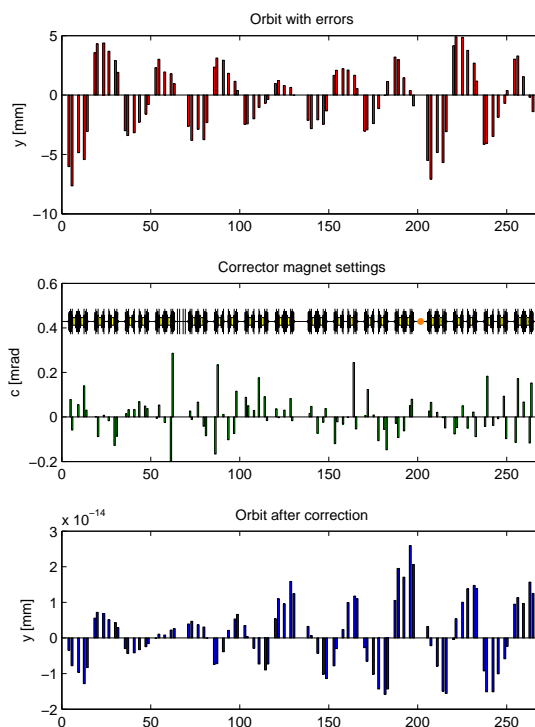


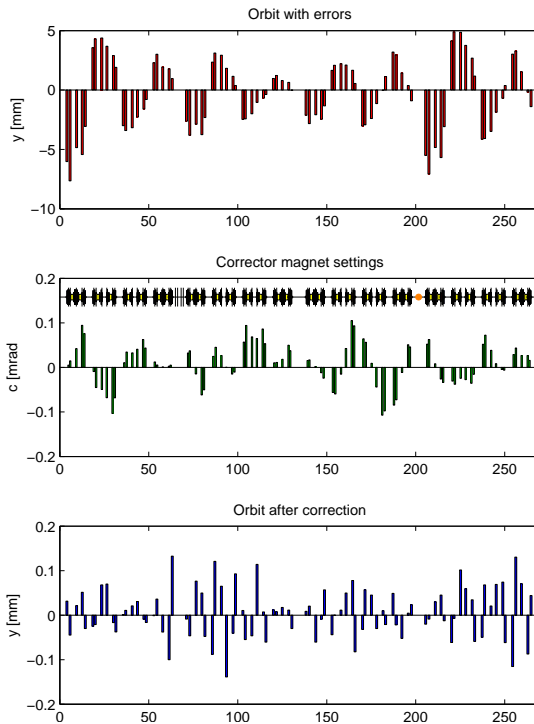
Figure: Example of correction for ALBA using response matrix.

- A vertical displacement of  $\sigma_y = 100 \mu\text{m}$  has been applied to all the elements of the ALBA machine.
- The resulting orbit is larger than 5 mm in the vertical plane.
- That will bring the trajectory of the electrons out of the vacuum chamber.
- Using SVD and all the  $w$ , the orbit has been corrected down to zero.
- But this requires a lot of the available corrector strength.



# Example for ALBA - II

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- A vertical displacement of  $\sigma_y = 100 \mu\text{m}$  has been applied to all the elements of the ALBA machine (same as before).
- Using SVD, but now imposing  $\varepsilon = 10^{-2}$ , the orbit has been corrected down to  $100 \mu\text{m}$ .
- This requires less than half of the correction power.

Figure: Example of correction for ALBA using response matrix.

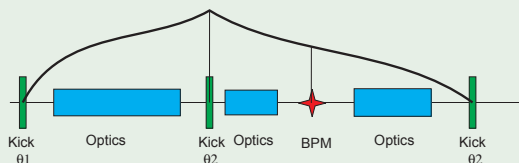


# Local Bumps

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- This method is based in an almost local compensation.
- It uses groups of correctors to create a local compensation of the reading of a BPM.
- Several options:  $\pi$ -bump; 3 or 4 correctors bumps...

## 3 corrector bump



The parameters for the bump are, in function of  $\theta_1$ :

$$\theta_2 = -\frac{\sqrt{\beta_1} \sin(\mu_{1,3})}{\sqrt{\beta_2} \sin(\mu_{2,3})} \theta_1$$

$$\theta_3 = \frac{\sqrt{\beta_1} \sin(\mu_{1,2})}{\sqrt{\beta_3} \sin(\mu_{2,3})} \theta_1$$

$$u_2 = \sqrt{\beta_1 \beta_2} \sin(\phi_2 - \phi_1) \theta_1$$

While the SVD method is the more used now (at least in synchrotron light sources), several other methods exists:

**Harmonic filtering** : consist in performing a Fourier analysis of the closed orbit. Then a small number of corrector are used to compensate the main components of the spectrum, that peaks around the tune.

**Least square fit** :Perfor a least square fit of the response matrix (equivalent to SVD).

**Most effective corrector** :.The orbit is perturbed by a few large perturbations (the most effective corrector). The most used method is the **MICADO** one. This methods can be equivalent to the SVD using only a few eigenvalues, but are more appropriate for the cases when we have a large number of correctors and/or BPMs.

**Movement of magnets** : It can be possible to move the magnet to compensate for orbit distortions.

## Effect of quadrupolar errors

The quadrupolar errors will affect the optical functions and the tunes of the machine

In the same way that we have learned that they are extra dipolar field affecting the trajectory of the particles, they are also extra quadrupolar fields affecting the designed optical functions and tunes of the particles.

The main sources of quadrupolar errors are:

- Differences between the theoretical values of the gradient of the quadrupoles and the real ones, due to fabrication errors, wrong settings of the power supplies or remanent fields.
- Quadrupoles fields due to the errors in the dipole magnets.
- Quadrupoles fields due to misalignment of the sextupoles respect to the reference orbit.
- Other components, like insertion devices.

A quadrupole error will produce a change in the focusing force  $K(s)$  of equation 4. As in the dipolar error, we will treat first a single gradient perturbation localized in a single point. In this case, a gradient error is treated as a thin lens, with focal length:

$$\frac{1}{f} = \Delta k \cdot L = \frac{q}{p_0} \Delta B' L \quad (55)$$

We can use the transfer matrix to determine the effect on the tune. The transfer matrix of the pertubated system will be the multiplication of the unperturbed system by the one of the quadrupole with focal length  $1/f = \Delta KL$ :

$$\mathbf{M}(s_0 \rightarrow C + s_0)_{\text{perturbed}} = \begin{pmatrix} 1 & 0 \\ -\Delta KL & 1 \end{pmatrix} \mathbf{M}(s_0 \rightarrow C + s_0)_{\text{ideal}} \quad (56)$$

$$\begin{pmatrix} \cos(2\pi Q) + \alpha(s_0) \sin(2\pi Q) & \beta(s_0) \sin(2\pi Q) \\ \gamma(s_0) \sin(2\pi Q) & \cos(2\pi Q) - \alpha(s_0) \sin(2\pi Q) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\Delta KL & 1 \end{pmatrix} \times \quad (57)$$

$$\begin{pmatrix} \cos(2\pi Q_0) + \alpha_0(s_0) \sin(2\pi Q_0) & \beta_0(s_0) \sin(2\pi Q_0) \\ \gamma_0(s_0) \sin(2\pi Q_0) & \cos(2\pi Q_0) - \alpha_0(s_0) \sin(2\pi Q_0) \end{pmatrix}$$

where  $s_0$  is the location of the error and the subscript 0 refers to unperturbed optics.

If we multiply the matrices and equate the trace of both sides, in order to get the tune, we get:

$$\cos(2\pi Q) = \cos(2\pi Q_0) \cos(2\pi Q) - \sin(2\pi Q_0) \sin(2\pi Q_0) \quad (58)$$

$$= \cos(2\pi Q_0) - \frac{\beta_0(s_0) \Delta KL \sin(2\pi Q_0)}{2} \quad (59)$$

## Tune change and stability

Assuming that the tune change is small, the previous equation give us:

$$\text{Tune change } \Delta Q = Q - Q_0$$

$$\Delta Q = \frac{1}{4\pi} \beta_0(s_0) \Delta KL \quad (60)$$

Equation 60 is only valid to the first order in  $\Delta K$ , since we have assumed that the optical functions were not perturbed. It also depends in that the lattice is stable after the perturbation, requiring that:

$$|\cos(2\pi Q)| = \left| \cos(2\pi Q_0) - \frac{\beta_0(s_0) \Delta KL \sin(2\pi Q_0)}{2} \right| < 1 \quad (61)$$

If the unperturbed tune  $Q_0$  was close to a half integer (of the form  $N + M/2$ ), the the value of  $\cos(2\pi Q_0)$  is small, and any perturbation can make the machine unstable, breaking the stability condition. For a given perturbation  $\Delta KL$ , there is a range of tunes around the half integer tune that make the motion unstable. This range is called the *half-integer stopband*, and the value is:

$$\Delta Q_{\text{stop}} = \frac{\beta_0(s_0)\Delta KL}{4\pi} \quad (62)$$

### Half integer resonance

Tunes close to a half integer should be avoided.



If we have many sources of errors  $\Delta KL_i$  in the machine, then the tune shift is:

$$\Delta Q = \frac{1}{4\pi} \sum_i \beta_0(s_i)\Delta(KL)_i \quad (63)$$

or for a continuous distribution of errors:

$$\Delta Q = \frac{1}{4\pi} \oint \beta_0(\tau)\delta K(\tau) d\tau \quad (64)$$

In this case, is more difficult to give a formula for the stopband, as the phase advance between errors should also be included.



Our next step is to evaluate the effect of one of the errors in the optical functions. If we write again the perturbed  $\mathbf{M}$  as in equation 56, but this time at an arbitrary point  $s$ , in function of the perturbation  $\Delta\text{KL}$  and of the unperturbed one:

$$\mathbf{M}(s \rightarrow s+C) = \mathbf{M}_0(s_0 \rightarrow s+C) \times \begin{pmatrix} 1 & 0 \\ -\Delta\text{KL} & 1 \end{pmatrix} \times \mathbf{M}_0(s \rightarrow s_0) \quad (65)$$

then, performing the multiplication and equating with the right hand side element  $m_{12}$ , we obtained:

$$m_{12} = \beta(s) \sin(2\pi Q) = (\beta_0(s) + \Delta\beta(s)) \sin(2\pi(Q_0 + \Delta Q)) \quad (66)$$

and from the value of  $\Delta Q$ , then:

$$\frac{\Delta\beta(s)}{\beta_0(s)} = -\frac{\Delta\text{KL}\beta_0(s_0)}{2\sin(2\pi Q_0)} \cos(2(|\phi_0(s) - \phi_0(s_0)| - \pi Q)) \quad (67)$$

And in the case of a distribution  $\delta K(s)$  of errors around the machine:

$$\Delta\beta(s) = -\frac{\beta_0(s)}{2\sin(2\pi Q_0)} \oint \beta_0(\tau) \delta K(\tau) \cos(2(|\phi_0(s) - \phi_0(s_0)| - \pi Q)) d\tau \quad (68)$$

The beta function oscillates with twice the betatron frequency, and from the previous formula is easy to see the half integer resonance.

The main sources of quadrupolar errors are:

- Error in the settings of quadrupoles (due either by error in the settings, construction or others). In this case, the error will be the  $\Delta K$  of the error.
- Misalignments of the sextupoles. The field in a sextupole magnet can be written as:

$$B_x = 2b_3 \frac{xy}{r_0^2}, \quad B_y = b_3 \frac{(x^2 - y^2)}{r_0^2} \quad (69)$$

in that case, an horizontal misalignment  $\Delta x$ , changes the field to:

$$B_x = 2b_3 \frac{xy}{r_0^2} - 2b_3 \frac{\Delta x}{r_0} \frac{y}{r_0} \quad (70)$$

$$B_y = b_3 \frac{(x^2 - y^2)}{r_0^2} - 2b_3 \frac{\Delta x}{r_0} \frac{x}{r_0} + b_3 \frac{\Delta x^2}{r_0^2} \quad (71)$$

In the case of a error in the rotation around  $s$   $\Delta\phi$  of one quadrupole, we are introducing a **skew** quadrupolar field, with value:

$$\Delta K_{skew} = K\Delta\phi \quad (72)$$

The effect of this components is to introduce a coupling between the motion of the horizontal and vertical plane. For a good performance of either a collider or a synchrotron light source, this is one of the key parameters to control.

For example, in a synchrotron light source, the vertical beam size (one of the key parameters) is determined by the vertical emittance. In absence of coupling, the value of  $\varepsilon_y$  is orders of magnitudes below the horizontal one. Once we introduce the coupling, this value will be given by:

$$\varepsilon_u = \frac{\chi}{1 + \chi} \varepsilon_x \quad (73)$$

In modern machines,  $\chi$  is in the range of 0.1% to 1%.

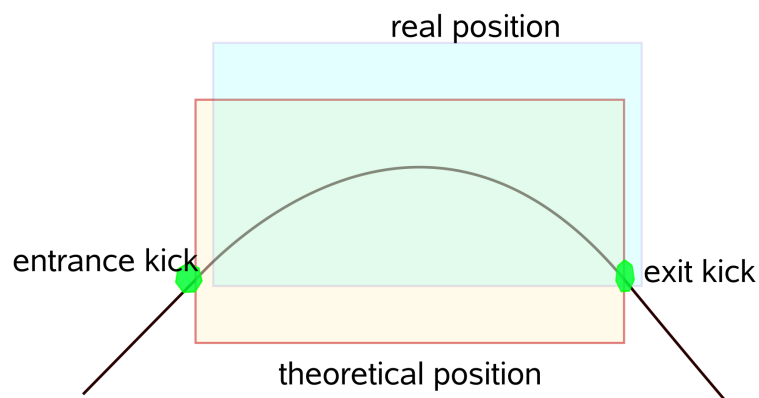
# A simple tutorial

In order to provide a better understanding of the effect of dipolar errors, we will examine the 2 kick representation method of dipolar imperfection.

# Overview

## Theorem

*A dipolar imperfection in a finite length element can be simulated by two thin kicks, one at the entrance, and one at the exit of it.*



This is a practical way to implement misalignment errors, or to simulate the effect of complex elements, as in the case of the insertion devices.

For simplify, we will extend the transfer matrix notation to include the effect of the kick (in a similar way as it was done to include dispersion):

$$\begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_{\text{exit}} = \begin{pmatrix} m_{1,1} & m_{1,2} & 0 \\ m_{2,1} & m_{2,2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_{\text{input}} \quad (74)$$

And the extended transfer matrix for a kick  $\theta$  is:

$$\begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_{\text{exit}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} u \\ u' \\ 1 \end{pmatrix}_{\text{input}} \quad (75)$$

And the transfer matrix for the system of the displaced element and the two kicks is:

$$\mathbf{M}_{2k} = \mathbf{M}_{k2} \times \mathbf{M}_{\text{element}} \times \mathbf{M}_{k1} \quad (76)$$

$$= \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,2}\theta_1 \\ m_{2,1} & m_{2,2} & m_{2,2}\theta_1 + \theta_2 \\ 0 & 0 & 1 \end{pmatrix} \quad (77)$$

This matrix has to be compared with the ones that we will obtain doing the change of the coordinate system, and from there we can identify the values of the kicks.

# Example 1: Displacement in s of one dipole

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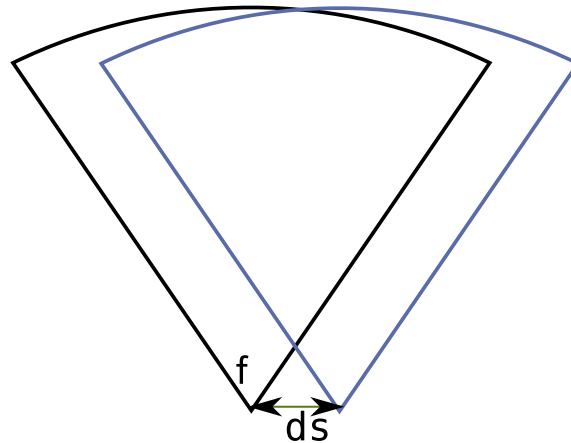
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The equations for the change of coordinates are:

$$X_i = x_i + dx \quad X'_i = x' = i \quad (78)$$

$$x_e = X_e + dx \quad X'_e = x' = e \quad (79)$$

$$(80)$$

and  $dx = ds \sin(\phi/2)$



Using this change of coordinates, we can write the transfer matrix as:

$$\mathbf{M}_{\text{disp}} = \begin{pmatrix} 1 & 0 & dX \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \mathbf{M}_{\text{element}} \times \begin{pmatrix} 1 & 0 & dX \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (81)$$

$$= \begin{pmatrix} m_{1,1} & m_{1,2} & (1 + m_{1,1})dx \\ m_{2,1} & m_{2,2} & m_{2,1}dx \\ 0 & 0 & 1 \end{pmatrix} \quad (82)$$

$$= \mathbf{M}_{2k} \quad (83)$$

$$= \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,2}\theta_1 \\ m_{2,1} & m_{2,2} & m_{2,2}\theta_1 + \theta_2 \\ 0 & 0 & 1 \end{pmatrix} \quad (84)$$

and the equivalent kicks are:

$$\theta_1 = -\theta_2 = \frac{ds}{\rho} \cos \frac{\phi}{2} \quad (85)$$



# Displacement dx in one focusing quadrupole

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In this case, the change of coordinates is:

$$X_i = x_i - dx \quad X'_i = x' = i \quad (86)$$

$$x_e = X_e + dx \quad X'_e = x' = e \quad (87)$$

$$(88)$$

And reproducing the same procedure, we obtain:

$$\theta_1 = \theta_2 = \sqrt{k} \tan \left( \sqrt{k} \frac{L}{2} \right) dx \quad (89)$$

## Two kind of kicks

Comparing both case, we can see that for same kind of errors, the two kicks will compensate each other, at first order, as is the of the displacement ds of one bending. For some errors, will kick acts in the same direction, and they will superimpose each other.



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