Introduction to charged particle optics.

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ALBA-CELLS Beam Dynamics

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References

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- H. Wiedemann, Particle Accelerator Physics I, Springer, 1999.
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- R. Bartiloni Lectures:`http://www.adams-institute.ac.uk/graduate_lectures/riccardo_bartolini/`

Introduction

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List of lectures-I

Introduction. Basic Concepts: M. Muñoz — This lecture. Review of the concepts: motion of electrons in a magnetic field, transfer matrices, etc...

Linear Optics: D. Einfeld — Optical functions, tunes, etc...

Effect of Errors and Correction: M. Muñoz — Dipolar (orbit errors). Correction of orbit. Quadrupolar errors.

Longitudinal Motion: F. Perez — Longitudinal motion due to the RF.

List of lectures-II

Collective Effect: T. Guenzel Effects due to the interaction between the electrons and between them and the vacuum chamber.

Non linear optics: G. Benedetti Introduction to the noon linear dynamics due to the sextupoles. Dynamic aperture, tune shift, etc...

Lifetime/Effect of IDs: Z. Martí Two parts: Lifetime of the electron beam, due to the interaction with the residual gas and scattering between electrons; Effect of the Insertion Devices in the dynamics of the electrons.

Lattice Design: D. Einfeld How to put everything together to desing a light source.

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Objective of the lecture

Objectives

The target of this lecture is to provide the basis of the linear motion of charged particles in electromagnetic fields, in particular in the longitudinal plane.
The emphasis will be put in relativistic particles moving in magnetic fields, reviewing the concepts of matricial optics, phase space and emittance.
The transfer matrices for the conventional building blocks of particle accelerators (dipoles and quadrupoles) are presented.

Some equations reminders

Lorentz's equation:

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{2.1}$$

- $\vec{F}$ is the electromagnetic force.
- $\vec{p}$ is the relativistic momentum.
- $\vec{v}$ is the relativistic velocity.
- $\vec{B}$ is the magnetic field vectors.
- $\vec{E}$ is the electric field vector.
- q is the electric charge.

Relativistic formulas

Total Energy

$$E_{tot}^2 = p^2c^2 + m_0^2c^4 = \left(T + m_0c^2\right)^2$$

where:

- E_{tot} is the total energy.
- T is the kinetic energy.
- m_0 is the rest mass.
- c is the speed of light.

Reduced velocity

$$\beta = \frac{v}{c} \tag{2.2}$$

Reduced energy

$$\gamma = \frac{E_{tot}}{m_0c^2} = \frac{1}{\sqrt{1-\beta^2}} \tag{2.3}$$

Job of the particle accelerator physicist

Job description

The basic description of the job of a particle accelerator physicist is to control the trajectory of charged particles (that can be electrons, positrons, protons, ions or more exotic (muons)) inside a particle accelerator system (either a synchrotron light source, a collider, a betatron, a linear accelerator or a simple transfer line). For that we have to:

- Control the energy of the particles (acceleration). This is the job of the RF system and the accelerating structures.
- Control the trajectory of the particles. This requires several components:
  - Guide the particles along the design path. This is the job of the dipoles.
  - Keep the particles inside the vacuum pipe (focusing of the particles). This is the job of the quadrupoles
  - Compensate for possible errors in the magnetic fields and imperfections. This is the job of the correctors, sextupoles and other magnets.

Equations of motion

Equation of motion

The basic equation of motion of a[†] charged particle in a electromagnetic field is the Lorentz's equation:

$$\vec{F} = \frac{d\vec{p}}{dt} = q\left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{3.1}$$

An option to solve the motion of the particles is to integrate numerically this equation. However this is very time consuming, and does not give us any of the global properties of the system, or help us to design the lattice for a workable particle accelerator.

[†] We will be dealing with single particle dynamics. No interaction between particles.

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Acceleration

From the definition of the relativistic momentum:

$$\vec{p} = m_0\gamma\vec{v}$$

the acceleration is given by:

$$\begin{aligned}
\frac{d\vec{p}}{dt} &= \dot{p} = \frac{d(m_0\gamma\vec{v})}{dt} \\
&= m_0\gamma\dot{\vec{v}} + m_0\dot{\gamma}\vec{v} \\
&= m_0\left(\gamma\dot{\vec{v}} + \gamma^3\beta\vec{v}\dot{v}/c\right) \\
&= \dot{\vec{p}}_\perp + \dot{\vec{p}}_{||}
\end{aligned}$$

where we have used the relation $\dot{\gamma} = \gamma^3\dot{v}\beta/c$.

The perpendicular force is:

$$\dot{\vec{p}}_\perp = m_0\gamma\dot{\vec{v}}_\perp$$

And the parallel force is:

$$\dot{\vec{p}}_{||} = m_0\gamma^3\dot{\vec{v}}_{||}$$

For relativistic particles ($\gamma >> 1$) the parallel acceleration is much more effective.

Electric and magnetic field efficiency

- It can be show that electric fiels are the most efficient to accelerate particles. The change in the kinetic energy is given by:

$$\Delta T = \int \vec{F} d\vec{s} = q \int \vec{E} d\vec{s} + \cancel{q \int \left(\vec{v} \times \vec{B}\right) \vec{v} dt}$$

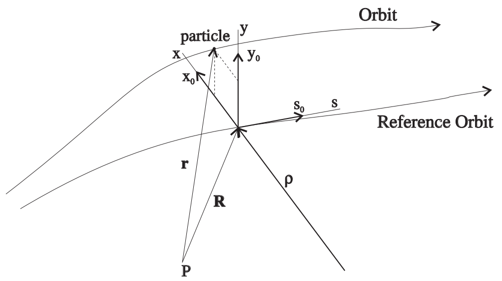
  i.e. electric fields are used for accelerating particles (RF cavities, etc). This subject will be ignored in this lecture.
- For a particle moving in the $\vec{z}$ direction, the $\vec{x}$ deviation is given by:

$$\frac{dp_x}{dt} = \vec{F}_x = q(E_x - v_z B_y)$$

- In general, we are dealing with relativistic particles and $v_z \approx c$, so magnetic fields are much more effective (a magnetic field of 1 Tesla correspond to an electric one of $3 \times 10^8 V/m$

The Curved coordinate system

- The cartesian coordinate system is not the most appropriate to describe the motion of particles in an accelerator.
- The selected coordinate system is the Frenet reference frame (also called the moving curved coordinated frame).



- It follows the ideal path of the particles along the accelerator.
- The curvature vector is defined as:

$$\vec{\kappa} = -\frac{d^2\vec{s}}{ds} \approx \frac{1}{\rho}$$

Ideal path

Now, from the Lorentz's equation we can obtain the equation for the ideal path:

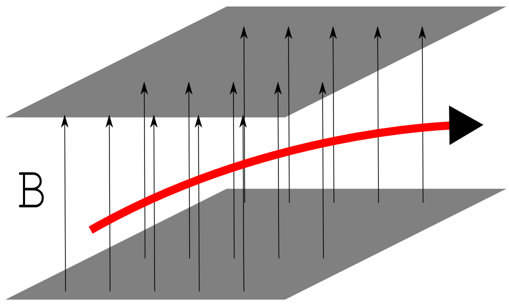
$$\frac{d\vec{p}}{dt} = m_0\gamma\frac{d^2\vec{s}}{dt^2} = m_0\gamma v_s^2\frac{d^2\vec{s}}{ds^2} = -m_0\gamma v_s^2\vec{\kappa} = q\left|\vec{v}\times\vec{B}\right|$$

and

$$\vec{\kappa} = -\frac{q}{p}\left|\frac{\vec{v}}{v_s}\times\vec{B}\right|$$

A simple case, a pure dipolar field

- Let's consider a simple case: the motion of a particle in an uniform magnetic field $\vec{B}$ perpendicular to the motion of the particle, with a longitudinal speed v_s close c (in that case $v_{x,y} \ll v_s$)



- In that case, we have the following equation for the radius of curvature:

$$\frac{1}{\rho} = |\kappa| = \left|\frac{q}{p}B\right| = \left|\frac{q}{\beta E_{tot}}B\right|$$

- The **magnetic rigidity** is defined as:

$$|B\rho| = \frac{p}{q}$$

In practical units, for the electron case:

$$\beta E_{tot}\,[\mathrm{GeV}] = 0.2998\,|B\rho|\,[\mathrm{Tm}]$$

Description of the magnetic field

We are going now to deriver the equation of motion of the particles in the curved rotating reference frame. For this, we will employ some assumptions

- The magnetic field is symmetric in the vertical plan, and $B_x(y=0) = B_s(y=0) = 0$ (flat accelerator). At any given s in the trajectory:

$$B_y(y) = B_y(-y);\ B_x(y) = -B_x(-y);\ B_s(y) = -B_s(-y)$$

- The field then can expanded as:

$$B_y = \sum_{i,k=0}^{\infty} y^{2i} x^k a_{ik}(s) \qquad \text{(even in y)}$$

$$B_x = y \sum_{i,k=0}^{\infty} y^{2i} x^k b_{ik}(s) \qquad \text{(odd in y)}$$

$$B_s = y \sum_{i,k=0}^{\infty} y^{2i} x^k d_{ik}(s) \qquad \text{(odd in y)}$$

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Maxwell's equation

Obviously, the magnetic field should obey the Maxwell's equation. In the curved coordinated system, and in abscence of time dependent fields or electrical currents, these are:

$$\nabla \times \vec{B} = \left(\frac{\rho}{\rho + x}\frac{\partial B_x}{\partial s} - \frac{1}{\rho + x}B_s - \frac{\partial B_s}{\partial x};\right.$$
$$\frac{\partial B_s}{\partial y} - \frac{\rho}{\rho + x}\frac{\partial B_y}{\partial s};$$
$$\left.\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = (0;\ 0;\ 0)$$

$$\nabla \cdot \vec{B} = \frac{\partial B_y}{\partial y} + \frac{\partial B_x}{\partial x} + \frac{\rho}{\rho + x}\frac{\partial B_s}{\partial s} + \frac{1}{\rho + x} = 0$$

This conditions provides us with a recursion formula for the $(a, b, c)_{i,k}$ coefficients.

Simplified magnetic field formula

Using the previous formulas, is possible to reach the following expression of the magnetic field in the simmetry plane:

$$B_y(s) = \frac{p}{q}\left(h(s) + k(s)x + \frac{1}{2}m(s)x^2 + \frac{1}{6}nx^3 + \mathbf{O}(4)\right)$$

where:

$$h = \frac{q}{p}B_y = \frac{1}{\rho} \quad \text{is the dipole compenent}$$

$$k = \frac{q}{p}\frac{\partial B_y}{\partial x} \quad \text{is the quadrupole compenent}$$

$$m = \frac{q}{p}\frac{\partial^2 B_y}{\partial x^2} \quad \text{is the sextupolar compenent}$$

$$n = \frac{q}{p}\frac{\partial^3 B_y}{\partial x^3} \quad \text{is the octupolar compenent}$$

This expressions are our first introduction to the multipolar decomposition of the field.

Magnetic field expansion

The general field expansion with symmetry plane is:

$$\begin{aligned}
\frac{q}{p}B_x &= k\,y + m\,x\,y + \frac{1}{2}n\,x^2\,y - \\
&\quad \frac{1}{6}\left(h\left(b - 2m\right) + a'' + n\right)x\,y^2 + \mathbf{O}(4) \\
\frac{q}{p}B_y &= h + k\,x + \frac{1}{2}m\,x^2 - \\
&\quad \frac{1}{2}b\,y^2 + \frac{1}{6}n\,x^3 - \frac{1}{2}\left(h\left(b - 2m\right) + a'' + n\right)x\,y^2 + \mathbf{O}(4) \\
\frac{q}{p}B_s &= h'\,y + a'\,x\,y + \left(h\,a' + \frac{1}{2}m'\right)x^2\,y + \mathbf{O}(4)
\end{aligned}$$

where

$$\begin{aligned}
a &= \tfrac{1}{2}h^2 + k \\
b &= h'' - h\,k + m
\end{aligned}$$

Magnetic Multipolar Expansion

- Is possible (deriving from Gaus laws, and choosing the righ gauge), to write the magnetic field as:

$$\begin{aligned} B_x + iB_y &= -\frac{\partial}{\partial x}\left(A_s(x,y) + iV(x,y)\right) \\ &= -\sum_{n=1}^{\infty} n\left(\lambda_n + i\mu_n\right)\left(x+iy\right)^{n-1} \end{aligned}$$

- Setting $b_n = -n\lambda_n$ and $a_n = n\mu_n$, we have

$$B_x + iB_y = \sum_{n=1}^{\infty}(b_n - ia_n)(x-iy)^{n-1} \tag{3.2}$$

- b_n are the normal coefficient of the field and a_n the skew.
- Careful with the US notation and the european.

Magnetic Multipolar Expansion

- When dealing with magnetic measurement, is custom to work with the following normalized units:

$$\begin{aligned} B_n &= \frac{b_n}{10^{-4}B_{main}}\rho^{n-1} \\ A_n &= \frac{a_n}{10^{-4}B_{main}}\rho^{n-1} \end{aligned}$$

- where B_{main} is the main component of the field (B_2 for a sextupole, for example), and B_n and A_n are the contribution of the multipoles at a reference radius ρ.
- The field is now expressed as:

$$B_x + iB_y = 10^{-4}B_{main}\sum_{n=1}^{\infty}(B_n - iA_n)\left(\frac{x-iy}{\rho}\right)^{n-1} \tag{3.3}$$

Linear equation of motion

The next step is to find the trajectory equations. In this coordinate system the time derivatives of the moving axes of the coordinate system are:

$$\begin{aligned} \dot{\vec{x}}_0 &= \frac{\dot{s}}{\rho}\vec{s}_0 \\ \dot{\vec{y}}_0 &= 0 \\ \dot{\vec{s}}_0 &= -\frac{\dot{s}}{\rho}\vec{x}_0 \end{aligned}$$

where $\dot{s} = ds/dt$ is the velocity projection on the reference particle orbit. The position and velocity of the particle in a fixed coordinate system P is:

$$\begin{aligned} \vec{r} &= x\vec{x}_0 + y\vec{y}_0 + \vec{R} \\ \dot{\vec{r}} &= \dot{x}\vec{x}_0 + x\dot{\vec{x}}_0 + \dot{y}\vec{y}_0 + \dot{s}\vec{s}_0 \end{aligned}$$

where $\dot{\mathbf{R}}$ has been replaced by $\dot{s}\,\mathbf{s_0}$.

Linear equation of motion

Identifying $\dot{\vec{r}}$ with $\vec{v}$, we obtain:

$$\begin{aligned} \vec{v} &= \dot{\vec{r}} \\ &= \dot{x}\vec{x}_0 + \dot{y}\vec{y}_0 + \left(1 + \frac{x}{\rho}\right)\dot{s}\vec{s}_0 \\ \dot{\vec{v}} &= \ddot{y}\vec{y}_0 + \left(\ddot{x} - \frac{\dot{s}^2}{\rho}\left(1 + \frac{x}{\rho}\right)\right) + \left(\left(1 + \frac{x}{\rho}\right)\ddot{s} + \frac{\dot{s}\dot{x}}{\rho}\right)\vec{s}_0 \end{aligned}$$

Replacing now the time variable t by the arc lenght s[†]

$$\begin{aligned} \dot{x} &= x'\dot{s} \\ \ddot{x} &= x''\dot{s}^2 + x' + \ddot{s} \\ \dot{y} &= y'\dot{s} \\ \ddot{y} &= y''\dot{s}^2 + y' + \ddot{s} \end{aligned}$$

† $\frac{d\xi}{dt} = \dot{\xi}$, $\frac{d\xi}{ds} = \xi'$

Linear equation of motion

Without electrical component, the Lorentz's equation is:

$$m\frac{d\vec{v}}{dt} = q\left(\vec{v} \times \vec{B}\right)$$

and replacing inside it the previous equation, we can obtain the following equation for the trajectories:

$$\begin{aligned} x'' + \frac{\ddot{s}}{\dot{s}^2}x' - \frac{1}{\rho}\left(1 + \frac{x}{\rho}\right) &= x - \frac{v}{\dot{s}}qp\left(y' B_s - \left(1 + \frac{x}{\rho}\right) B_y\right) \\ y'' + \frac{\ddot{s}}{\dot{s}^2}y' &= \frac{v}{\dot{s}}qp\left(x' B_s - \left(1 + \frac{x}{\rho}\right) B_x\right) \end{aligned} \quad (3.4)$$

Simplifiying hypothesis

In order to solve the previous equation, the following hypothesis are use:

- No longitudinal component of the magnetic field, $B_s = 0$. Transition areas at the end of the magnetic elements are ignored, even if doing this breaks Maxwell's Laws.
- Only linear components of the field: The magnets only have dipolar and quadrupolar components.
- Small angle amplitude movements (paraxial approximation). The transversal velocities $\dot{x}$, $\dot{y}$ are considered to be much smaller than the longitudinal one, $\dot{s}$, which is close to c.
- There is not coupling between the motion in the two transversal plane. No skew quadrupoles.

Approximations

The previous hipothesys allows us to perfrom the following aproximations:

$$\begin{aligned}
\frac{v}{\dot{s}} &= \sqrt{\left(1+\frac{x}{\rho}\right)^2 + x''^2 + y''^2} \\
&\approx 1+\frac{x}{\rho} \\
\frac{\ddot{s}}{\dot{s}^2} &= \frac{d}{dt}\frac{v}{\dot{s}} \\
&\approx 0 \\
\frac{1}{p} &\approx \frac{1}{p_0}\left(1-\frac{\Delta p}{p_0}\right) \\
\frac{q}{p}B_x &\approx k\,y \\
\frac{q}{p}B_y &\approx -\frac{1}{\rho}+k\,x \\
\frac{q}{p}B_s &\approx 0
\end{aligned}$$

Simplified equation of motion

After applying the previous simplifications, the equation of motion (3.4) is symplified to:

Equation of motion

$$\begin{aligned}
x'' - \left(k(s) - \frac{1}{\rho^2}\right)x &= \frac{1}{\rho}\frac{\Delta p}{p_0} \\
y'' + k(s)\,y &= 0
\end{aligned} \qquad (3.5)$$

In the next section, we will review the matricial method to solve the equation of motion.

Matricial Optics

Reminder of basic equations of motion

In previous lectures, you have learned that in the moving coordinate system, the Lorentz equation

$$\frac{d}{dt}\vec{v} = \frac{q}{m}\left(\vec{v} \times \vec{B}\right)$$

becomes the following two uncoupled equations:

$$\frac{d^2x}{ds^2} - \left(k(s) - \frac{1}{\rho^2}\right)x = \frac{1}{\rho}\frac{\Delta p}{p_0} \tag{4.1}$$

$$\frac{d^2y}{ds^2} + k(s)y = 0 \tag{4.2}$$

where:

$$k(s) = \frac{1}{B\rho}\frac{\partial B_y(s)}{\partial x}, \text{ focusing strength}$$

ρ is the radius of curvature of the particles in the magnet

$\frac{\Delta p}{p_0}$ is the momentum deviation respect the reference particle

If we concentrate in the on-energy particle ($p = p_0$), both equations 4.1 and 4.2 became homogenous and can be written as:

$$u'' + K(s)u = 0 \tag{4.3}$$

where u stands for x or y and $K(s)$ is given by:

$$K(s) = \begin{cases} -\left(k(s) - \frac{1}{\rho^2}\right) & u = x \\ k(s) & u = y \end{cases} \tag{4.4}$$

From here we can see that is difficult to focus simultaneously in both planes

Linear Equation of motion

Equation (4.3) is the equation of an anharmonic oscillator (Hill's equation).
If we write:

$$\vec{u} = \begin{pmatrix} u \\ u' \end{pmatrix}$$

then, the second order diferential equation becomes a system of two first order, that can be writen as:

$$\frac{d\vec{u}}{ds} + \begin{pmatrix} 0 & -1 \\ K(s) & 0 \end{pmatrix} \times \vec{u} = 0 \tag{4.5}$$

In the solution can be written as the linear combination of two independent particular solutions:

$$\vec{u}(s) = A\vec{u}_1(s) + B\vec{u}_2(s) \tag{4.6}$$

Principal solutions

We would choose two independent solutions with the following characteristics:

Cosine-line solution C : The inital conditions for this solution are:

$$\vec{u}_C(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.7}$$

Sine-line solution S : The inital conditions for this solution are:

$$\vec{u}_S(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{4.8}$$

And the solution for a given set of initial coordinates $\vec{u}_0 = (u_0; u'_0)$ is given by:

$$u(s) = u_0 C(s) + u'_0 S(s) \tag{4.9}$$

$$u'(s) = u_0 C'(s) + u'_0 S'(s) \tag{4.10}$$

Matricial form of the solution

Equation 4.9 can be writen in matricial form as:

$$\begin{pmatrix} u(s) \\ u'(s) \end{pmatrix} = \mathbf{M}(s_0 \mapsto s) \begin{pmatrix} u(s_0) \\ u'(s_0) \end{pmatrix} \tag{4.11}$$

where $\mathbf{M}(s_0 \mapsto s)$ is a 2×2 matrix given by:

$$\mathbf{M}(s_0 \mapsto s) = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \tag{4.12}$$

K constant: Harmonic Oscillator

If K is constant (for example inside a dipolar magnet, if we ignore the end field effect), then the principal solutions C and S are:

$$\vec{C}(s) = \begin{pmatrix} \sin(\sqrt{K}s) \\ \sqrt{K}\cos(\sqrt{K}s) \end{pmatrix} \tag{4.13}$$

$$\vec{S}(s) = \begin{pmatrix} \cos(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) \end{pmatrix} \tag{4.14}$$

and the transport map **M** is given by:

$$\mathbf{M}(s_0 \mapsto s) = \begin{pmatrix} \cos(\sqrt{K}(s-s_0)) & \sin(\sqrt{K}(s-s_0)) \\ -\sqrt{K}\sin(\sqrt{K}(s-s_0)) & \sqrt{K}\cos(\sqrt{K}(s-s_0)) \end{pmatrix}$$

- If K is positive we have focusing.
- If K is negative, we obtain the hyperbolic sinus and cosinus, and the particle is not focused.

Matricial optics

For a constant k, the solution of (4.3) allows the use of a matrix $\mathbf{M}(s|s_0)$ as the transfer map between the initial conditions (s_0) of the particle and the exit conditions (s) as:

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \mathbf{M}(s|s_0) \times \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0} = \begin{pmatrix} C(s|s_0) & S(s|s_0) \\ C'(s|s_0) & S'(s|s_0) \end{pmatrix} \times \begin{pmatrix} u \\ u' \end{pmatrix}_{s_0}$$

Unit Jacobian

It can be shown that:

$$\det(\mathbf{M}) = CS' - C'S = 1$$

that is true for conservative systems.

Stable motion

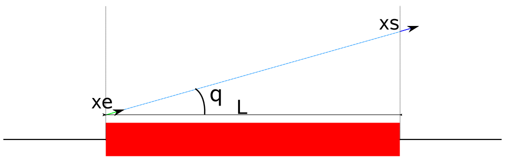
For a periodic system, the motion is stable only if the eigenvalues of **M** are on the unity circle, that is equivalent (for a 2×2 matrix) to:

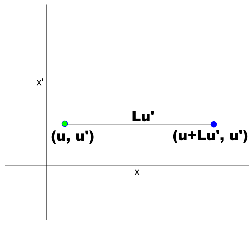
$$\left| \frac{1}{2}(\mathbf{M}_{11} + \mathbf{M}_{22}) \right| \leqslant 1$$

that is true for conservative systems.

Drift space

Let's consider the simplest example: A drift space (no magnetic element) of length L:





The input particle is: $x_e = (u, u')$.
The exit particle is: $x_s = (u_s, u_s') = (u + L \times u', u')$.
This can be written in matrix form as:

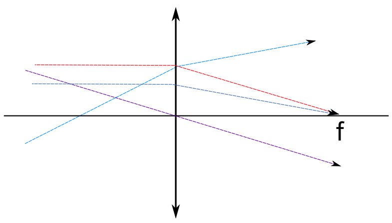
$$\begin{pmatrix} u_s \\ u_s' \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} u \\ u' \end{pmatrix}$$

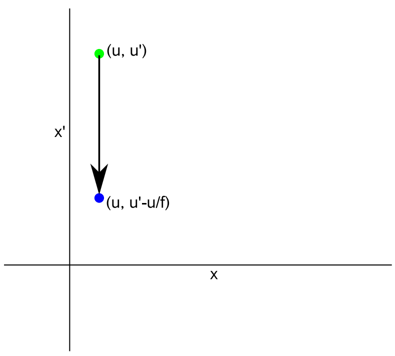
and the transfer matrix of a drift space of length L can be written as:

$$\mathbf{M}_{\mathtt{drift}}(L) = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Thin lens

The second simplest example: A thin lens of focal length f: a zero length element that changes the transversal momentum of the particles. This corresponds to the limit case for a quadrupole.





The input particle is: $x_e = (u, u')$.
The exit particle is:
$x_s = (u_s, u_s') = (u, u' - \frac{u}{f})$.
This can be written in matrix form as:

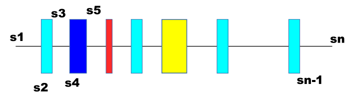
$$\begin{pmatrix} u_s \\ u_s' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \times \begin{pmatrix} u \\ u \end{pmatrix}$$

and the transfer matrix can be written as:

$$\mathbf{M}_{\mathtt{thin}}(L) = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

Piecewise solution

In the case that we have a system composed of drift spaces and thin lenses, it is easy to see that the transfer matrix of the whole system can be build from the transfer matrix of each of the elements:



$$\begin{aligned}\vec{u}_n &= \mathbf{M}(s_n|s_0) \times \vec{u}_0 \\ &= \mathbf{M}(s_n|s_1) \times (\mathbf{M}(s_1|s_0) \times \vec{u}_0) \\ &= (\mathbf{M}(s_n|s_1) \times \mathbf{M}(s_1|s_0)) \times \vec{u}_0\end{aligned}$$

$$= (\mathbf{M}(s_n|s_{n-1}) \times \mathbf{M}(s_{n-1}|s_{n-2}) \times \ldots \mathbf{M}(s_2|s_1) \times \mathbf{M}(s_1|s_0)) \times \vec{u}_0$$

Matrix composition

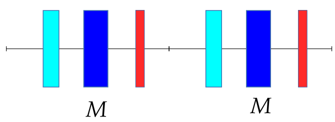
$$\mathbf{M}(s_n|s_0) = \mathbf{M}(s_n|s_{n-1}) \times \mathbf{M}(s_{n-1}|s_{n-2}) \times \ldots \mathbf{M}(s_2|s_1) \times \mathbf{M}(s_1|s_0)$$

Symmetric transport lines

We can examine two simple symmetries in a transport line:

- System with periodic symmetry

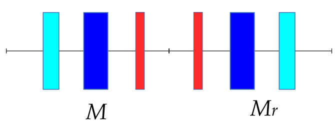
$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



$$\mathbf{M}_{tot} = \mathbf{M} \times \mathbf{M} = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$$

- System with mirror symmetry

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mathbf{M}_r = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$$



$$\mathbf{M}_{tot} = \mathbf{M}_r \times \mathbf{M} = \begin{pmatrix} ab + bc & 2db \\ 2ac & cb + ad \end{pmatrix}$$

4×4 matrices

- Until now we have treated both planes independently, in an abstract way.

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C_x & S_x \\ C'_x & S'_x \end{pmatrix} (s|s_0) \times \begin{pmatrix} x \\ x' \end{pmatrix}_0$$

$$\begin{pmatrix} y \\ y' \end{pmatrix}_s = \begin{pmatrix} C_y & S_y \\ C'_y & S'_y \end{pmatrix} (s|s_0) \times \begin{pmatrix} y \\ y' \end{pmatrix}_0$$

- It is possible to combine the 2×2 matrices of both planes in a single 4×4 matrix:

$$\begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_s = \begin{pmatrix} C_x & S_x & 0 & 0 \\ C'_x & S'_x & 0 & 0 \\ 0 & 0 & C_y & S_y \\ 0 & 0 & C'_y & S'_y \end{pmatrix} \times \begin{pmatrix} x \\ x' \\ y \\ y' \end{pmatrix}_0$$

The motion is uncoupled (one of our assumptions), so those elements are 0

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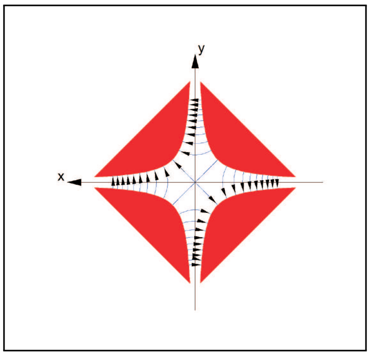
Matrices for real elements. Quadrupole

- The quadrupole is the more realistic case of the thin lens.
- The field inside is given by:

$$\vec{B} = (-Gy, -Gx, 0)$$

where G is the gradient ([T/m]). The normalized k is given by:

$$k = \frac{G}{B\rho}$$



The 2×2 matrix through a quad is:

$$\begin{pmatrix} u \\ u' \end{pmatrix}_s = \begin{pmatrix} \cos\left(\sqrt{k}(s-s_0)\right) & \frac{1}{\sqrt{k}} \sin\left(\sqrt{k}(s-s_0)\right) \\ -\sqrt{k}\sin\left(\sqrt{k}(s-s_0)\right) & \cos\left(\sqrt{k}(s-s_0)\right) \end{pmatrix} \times \begin{pmatrix} u \\ u' \end{pmatrix}_0$$

Focusing and defocusing quadrupole

- If $k > 0$, the quadrupole is focusing, and the matrix is:

$$\mathbf{M}_{QF} = \begin{pmatrix} \cos\left(\sqrt{k}L\right) & \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}L\right) \\ -\sqrt{k}\sin\left(\sqrt{k}L\right) & \cos\left(\sqrt{k}L\right) \end{pmatrix}$$

- If $k < 0$, the quadrupole is defocusing, and the matrix is:

$$\mathbf{M}_{QD} = \begin{pmatrix} \cosh\left(\sqrt{|k|}L\right) & \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}L\right) \\ \sqrt{|k|}\sinh\left(\sqrt{|k|}L\right) & \cosh\left(\sqrt{|k|}L\right) \end{pmatrix}$$

- by setting $\sqrt{|k|}L \to 0$, the matrices become:

$$\mathbf{M}_{QF,QD} = \begin{pmatrix} 1 & 0 \\ -kL & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$$

4×4 matrix for the quadrupole

- Notice that a quadrupole focusing in the horizontal plane is defocusing in the vertical, and viceversa.
- The 4×4 matrix for a horizontal focusing quadrupole is:

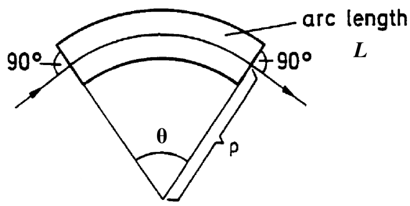
$$\mathbf{M}_{QFh} = \begin{pmatrix} \cos\left(\sqrt{k}L\right) & \frac{1}{\sqrt{k}}\sin\left(\sqrt{k}L\right) & 0 & 0 \\ -\sqrt{k}\sin\left(\sqrt{k}L\right) & \cos\left(\sqrt{k}L\right) & 0 & 0 \\ 0 & 0 & \cosh\left(\sqrt{|k|}L\right) & \frac{1}{\sqrt{|k|}}\sinh\left(\sqrt{|k|}L\right) \\ 0 & 0 & \sqrt{|k|}\sinh\left(\sqrt{|k|}L\right) & \cosh\left(\sqrt{|k|}L\right) \end{pmatrix}$$

- We need to find a solution if we want to focus simultanesly in both planes.

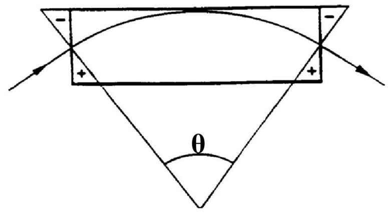
Dipoles

- The other linear element that we are considering is the dipole.
- A dipole where the input and exit faces are perpendicular to the ideal trajectory is know as a **sector dipole**.
- One where the faces are parallel is know as a **rectangular dipole**.

Let's consider first the sector dipole:



Sector bend



Rectangular bend

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Sector Dipole

- For a dipole of length L, bending radius ρ and deflecting angle $\theta = \frac{L}{\rho}$ and no quadrupole component in it, $k = \frac{1}{\rho^2}$, and the horizontal (assuming horizontal deflection usually) transfer matrix is:

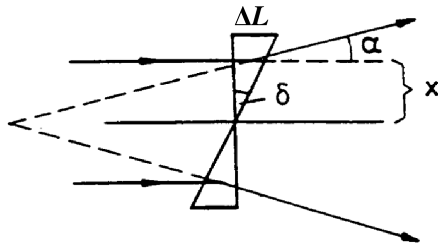
$$\mathbf{M}_{x,sbend} = \begin{pmatrix} \cos\theta & \rho\sin\theta \\ -\frac{1}{\rho}\sin\theta & \cos\theta \end{pmatrix}$$

- In the vertical plane, the matrix is the one of a drift space:

$$\mathbf{M}_{y,sbend} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}$$

Rectangular Dipole

- In that case, we have an effect when crossing the entrance an exit faces.
- The effect is equivalent to a thin lens wiht $\frac{1}{f} = \frac{\tan\delta}{\rho}$, acting in both planes.



- The transfer matrix for the edge is:

$$\mathbf{M}_{\mathtt{edge}} = \begin{pmatrix} 1 & 0 \\ -\frac{\tan\delta}{\rho} & 1 \end{pmatrix}$$

- The total transer matrix is $\mathbf{M}_{\mathtt{rbend}} = \mathbf{M}_{\mathtt{edge}} \times \mathbf{M}_{\mathtt{sbend}} \times \mathbf{M}_{\mathtt{edge}}$

$$\mathbf{M}_{x,\mathtt{rbend}} = \begin{pmatrix} 1 & \rho\sin\theta \\ 0 & 1 \end{pmatrix} ; \quad \mathbf{M}_{y,\mathtt{rbend}} = \begin{pmatrix} 1 - \frac{L}{f_y} & L \\ -\frac{2}{f_y} + \frac{2}{f_y^2} & 1 - \frac{L}{f_y} \end{pmatrix}$$

where $\frac{1}{f_y} = \frac{\tan\theta/2}{\rho}$

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Combined function magnet

- The most general magnet is one that combines dipole field and quadrupole one (sometimes know as synchrotron magnet).
- The transfer matrix (for a sector magnet) is:

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos\phi & \frac{1}{\sqrt{|K|}}\sin\phi \\ -\sqrt{|K|}\sin\phi & \cos\phi \end{pmatrix} \quad K > 0,\ QF$$

$$\begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \frac{1}{\sqrt{|K|}}\sinh\phi \\ \sqrt{|K|}\sinh\phi & \cosh\phi \end{pmatrix} \quad K < 0,\ QD$$

with

$$K = \begin{cases} -k + \frac{1}{\rho^2} & \text{in the } x \text{ direction} \\ k & \text{in the } y \text{ direction} \end{cases}$$

$$\phi = L\sqrt{|K|}$$

where L is the length of the magnet.

Off Energy Particles

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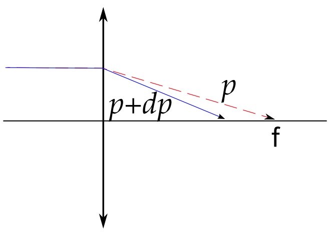
Off-energy term

- In equation (3.5) there is a inhomogeneous term $\left(\frac{1}{\rho}\frac{\Delta p}{p}\right)$ in the horizontal equation of motion. When solving the equation of motion, we have ignored it, concentrating in the on-energy particles.
- This term also affects the motion of the particles.
- For example, in a quadrupole, the focalization would be different:



The normalized quadrupole gradient is:

$$k = \frac{qG}{p_0} \tag{5.1}$$

for the off-momentum partice:

$$\Delta k = \frac{dK}{dp}\Delta p = -\frac{qG}{p_0}\frac{\Delta p}{p_0} = -\mathbf{k}\frac{\mathbf{\Delta p}}{\mathbf{p_0}} \tag{5.2}$$

Off-energy particles in a dipole.

- For a dipole, we have the equation of the magnetic rigidity:

$$B\rho = \frac{p_0}{q}$$

- For off-momentum particles, there is change in the bending radius:

$$B\left(\rho + \Delta\rho\right) = \frac{p_0 + \Delta p}{q}$$

- And from there is trivial to get:

$$\frac{\Delta\theta}{\theta} = -\frac{\Delta\rho}{\rho} = -\frac{\Delta p}{p_0}$$

- Off-momentum particles get a different deflection:

$$\Delta\theta = -\theta\frac{\Delta p}{p_0} \tag{5.3}$$

- This effect and the one in the quadrupole is equivalent to the effect of the optical elements (prism for bendings and lens to quadrupoles)

Equation of the dispersion.

- If we go now back to the horizontal Hill's equation (3.5):

$$x'' + k(s)x = \frac{1}{\rho(s)}\frac{\Delta p}{p_0} \tag{5.4}$$

- The general solution can be written as a combination of the solution of the homogeneous and inhomogeneous:

$$x(s) = x_H(s) + x_I(s) = x_H(s) + D(s)\frac{\Delta p}{p_0}$$

where $D(s)$ (the dispersion function) is a particular solution of the inhomogeneous equation for $\frac{\Delta p}{p_0} = 1$:

$$D''(s) + k_s(s)D = \frac{1}{\rho(s)} \tag{5.5}$$

and initial conditions:

$$\begin{pmatrix} D \\ D' \end{pmatrix}_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Using perturbation theory, is possible to show that dispersion function can be written in term of the principal trajectories C and S as:

$$D(s) = S(s)\int_0^s \frac{1}{\rho}C(\tau)d\tau - C(s)\int_0^s \frac{1}{\rho}S(\tau)d\tau$$

- The function satisfies equation (5.5):

$$\begin{aligned}
D' &= S'(s)\int_0^s \frac{1}{\rho}C(\tau)d\tau - C'(s)\int_0^s \frac{1}{\rho}S(\tau)d\tau \\
D'' &= S''(s)\int_0^s \frac{1}{\rho}C(\tau)d\tau - C''(s)\int_0^s \frac{1}{\rho}S(\tau)d\tau + \frac{1}{\rho}(CS' - SC') \\
&= S''(s)\int_0^s \frac{1}{\rho}C(\tau)d\tau - C''(s)\int_0^s \frac{1}{\rho}S(\tau)d\tau + \frac{1}{\rho} \\
&= -k\left(S(s)\int_0^s \frac{1}{\rho}C(\tau)d\tau - C(s)\int_0^s \frac{1}{\rho}S(\tau)d\tau\right) + \frac{1}{\rho} \\
&= -kD + \frac{1}{\rho}
\end{aligned}$$

Extended matrix for dispersion

- We can extend the matrix formalis to include the dispersion:
- From the expression of the total trajectory:

$$\begin{aligned}
x(s) &= x_H(s) + D(s)\tfrac{\Delta p}{p} &&= C(s)x_0 + S'(s)x_0' + D(s)\frac{\Delta p}{p} \\
x'(s) &= x_H'(s) + D'(s)\tfrac{\Delta p}{p} &&= C'(s)x_0 + S(s)x_0' + D'(s)\frac{\Delta p}{p}
\end{aligned}$$

- We can write:

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C(s) & S'(s) & D(s) \\ C'(s) & S'(s) & D'(s) \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0 = \mathbf{M}_{3\times 3} \times \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_0$$

Chromatic closed orbit

- In a periodic system (for example an storage ring), the on-energy particles oscillate around the design trajectory (closed orbit).
- The off-momentum particles will oscillate around the so-called chromatic closed orbit, different for each energy. For a given energy ($\frac{\Delta p}{p}$), this orbit is given by:

$$x_D = D_{per}(s)\frac{\Delta p}{p}$$

where $x_D = D_{per}(s)$ is the periodic solution for the dispersion, given by:

$$\begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} = \mathbf{M}_{3\times3}(s|s) \times \begin{pmatrix} D \\ D' \\ 1 \end{pmatrix} \tag{5.6}$$

where $\mathbf{M}_{3\times3}$ is the extened transfer matrix.

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3×3 Matrix for some elements

Quads and drift

$$\mathbf{M}_{3\times3} = \begin{pmatrix} \mathbf{M}_{2\times2} & . & 0 \\ . & . & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Sector bend

$$\mathbf{M}_{3\times3} = \begin{pmatrix} \mathbf{M}_{2\times2} & . & \rho(1-\cos\theta) \\ . & . & \sin\theta \\ 0 & 0 & 1 \end{pmatrix}$$

Edge focusing

$$\mathbf{M}_{3\times3} = \begin{pmatrix} \mathbf{M}_{2\times2} & . & 0 \\ . & . & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rectangular bend

$$\mathbf{M}_{3\times3} = \begin{pmatrix} \mathbf{M}_{2\times2} & . & \rho(1-\cos\theta) \\ . & . & 2\tan\frac{\theta}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

Combined function magnet

QF, K > 0

$$\mathbf{M}_{3\times 3} = \begin{pmatrix} \cos\phi & \frac{1}{\sqrt{|K|}}\sin\phi & \frac{1}{\rho K}(1-\cos\phi) \\ -\sqrt{|K|}\sin\phi & \cos\phi & \frac{\sin\phi}{\rho\sqrt{K}} \\ 0 & 0 & 1 \end{pmatrix}$$

QD, K > 0

$$\mathbf{M}_{3\times 3} = \begin{pmatrix} \cosh\phi & \frac{1}{\sqrt{|K|}}\sinh\phi & -\frac{1}{\rho|K|}(1-\cosh\phi) \\ \sqrt{|K|}\sinh\phi & \cosh\phi & \frac{\sin\phi}{\rho\sqrt{|K|}} \\ 0 & 0 & 1 \end{pmatrix}$$

$$K = \begin{cases} -k + \frac{1}{\rho^2} & x \text{ direction} \\ k & y \text{ direction} \end{cases}, \quad \phi = L\sqrt{|K|}$$

Momentum compaction factor

- Off-momentum particles travel a diferent orbit with a diferent lenght than the ideal one.
- The relative change of the path lenght with the relative momentum change is the so called momentum compaction factor α_p:

$$\alpha_p \equiv \frac{p}{C}\frac{dC}{dp} = \frac{\frac{\Delta C}{C}}{\frac{\Delta p}{p}} \tag{5.7}$$

- The change in the circumference is given by:

$$\Delta C = \oint D\frac{\Delta p}{p}d\theta = \oint D\frac{\Delta p}{p}\frac{ds}{\rho}$$

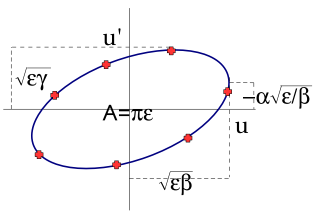
- So the total momentum compaction is:

$$\alpha_p = \frac{1}{C}\oint\frac{D(s)}{\rho(s)}ds \tag{5.8}$$

Emittance and phase space

The concept of emittance

- Until now we have studied only the motion of a single particle.
- A very usefull concept to relate the dynamics of a single particle and the one of a bunch of particles is the one of emittance (ε)



- Particles moving in a periodic stable linear system follows a closed trajectory in the phase space.
- This trajectory is an ellipse.
- The ellipse is transformed when moving along magnets.
- The area of the ellipse is constant (Liouville's theorem).

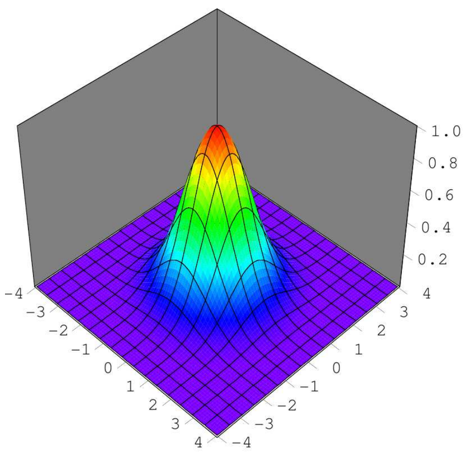
$$\gamma u^2 + 2\alpha u u' + \beta u'^2 = \varepsilon \tag{6.1}$$

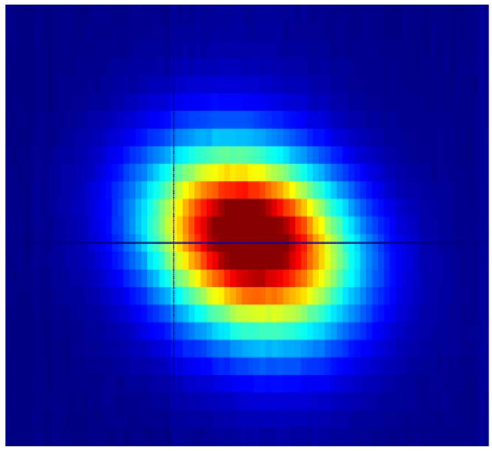
Emittance

The emittance is defined as $A = \pi\varepsilon$

Beam of particles

- In a real machine, the number of particles N in a beam is between millions and billions.
- The beam will be represented by a distribution of particles $f(\vec{u})$ in the phase space.





$$N = \int f(x, x', y, y')dxdx'dydy'$$

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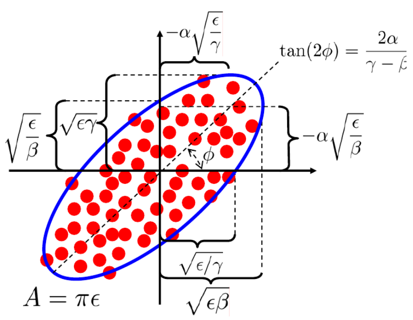
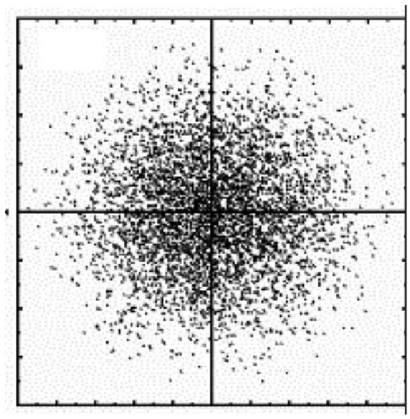
Emittance and beams of particles

- We can relate the emittance of a single particle with the area occupied by the distribution $f(\vec{u})$.
- For linear motion, and in absence of radiation, $f(\vec{u})$ must follow the Liouville's theorem.
- The area occupied by $f(\vec{u})$ in the phase space will be constant along the optical system.
- In general we will model the behaviour of the N particles by the distribution $f(x, x', y, y', /phi, E)$ in the 6D phase space.
- In general this distribution can be a "hard edge" constant distribution, a gaussian or similar.





Beam matrix

- In the same way that we have found an expresion to tranport the position of the particles around the system of magnets ($\vec{x}(s) = \mathbf{M}(s|s_0)\vec{x}_0$), we want to find one to tranport the beam ellipse around system.
- The general equation of an n-dimension ellipse is:

$$\vec{u}^{\top} \times \sigma^{-1} \times \vec{u} = \mathbf{I} \quad (6.2)$$

  where σ is n-dimension symmetric matrix.
- The volume of this ellipse is

$$V_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1+\frac{n}{2})}\sqrt{\det \sigma} \quad (6.3)$$

- For $n = 2$, equation (6.2) becomes:

$$\sigma_{1,1}x^2 + 2\sigma_{1,2}xx' + \sigma_{2,2}x'^2 = 1$$

- Comparing this last equation to (6.1), we can get the definition of the beam matrix:

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Beam matrix II

Beam matrix

$$\sigma = \begin{pmatrix} \sigma_{1,1} & \sigma_{1,2} \\ \sigma_{2,1} & \sigma_{2,2} \end{pmatrix} = \varepsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \quad (6.4)$$

The volume of the beam for this case is:

$$V_2 = \pi\sqrt{\det \sigma} = \pi\sqrt{\sigma_{1,1}\sigma_{2,2} - \sigma_{1,2}^2} = \pi\varepsilon$$

recovering our definiton of the emittance.

Transport of the beam matrix

- Let $\mathbf{M}$ be the transfer matrix from point s_0 to s_1, and x_0 and x_1 the position of the beam at those points ($\vec{x}_1 = \mathbf{M}\vec{x}_0,\ \vec{x}_0 = \mathbf{M}^{-1}\vec{x}_1$)
- Then:

$$\begin{aligned}
\vec{x}_1^\top \times \sigma_1^{-1} \times \vec{x}_1 &= 1 \\
\vec{x}_0^\top \times \sigma_0^{-1} \times \vec{x}_0 &= 1 \\
\left(\mathbf{M}^{-1}\vec{x}_1\right)^\top \times \sigma_0^{-1} \times \left(\mathbf{M}^{-1}\vec{x}_1\right) &= 1
\end{aligned}$$

after some matrix manipulation, and using the identity $\left(\mathbf{M}^\top\right)^{-1} \sigma_0^{-1} \left(\mathbf{M}\right)^{-1} = \left(\mathbf{M}\sigma_0\mathbf{M}^\top\right)^{-1}$, we obtain the equation for the transport of the beam matrix, using the transfer matrix $\mathbf{M}$:

Transport of the beam matrix

$$\sigma_1 = \mathbf{M}\sigma_0\mathbf{M}^\top$$

Matricial optics

Point to point imaging

- There is a point to point imaging between points $A = (x_a, x'_a)$ and $B = (x_b, x'_b)$ when any ray starting from A goes to B, independent of the initial angle x'_a.
- In this case, the component M_{12} of the system **M** is zero:

$$M_{12} = 0$$

- The magnification G of the system is defined as:

$$x_b = M_{11} x_a = G x_a$$
$$G = M_{11}$$

Focal Points

Let's consider an optical system with transfer matrix **M**.

- The object focal point is the point situated at a distance F_0 upstream, on axis ($x = 0$), such as any ray starting from fis point is parallel to the axis at the exit.
- In a similar way, we can define the image focal point.
- From the definition of the focal point, and using some matrix calculation:

$$\begin{pmatrix} x_s \\ 0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \times \begin{pmatrix} 1 & F_o \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ x_e \end{pmatrix} \quad (7.1)$$

leading to:

$$(M_{21} F_o + M_{22}) \, x'_e = 0 \quad (7.2)$$

and to the value to the object focal point:

$$F_o = -\frac{M_{22}}{M_{21}} \quad (7.3)$$

- and similarly for the image focal point:

$$F_i = -\frac{M_{11}}{M_{21}} \quad (7.4)$$

Principal planes

The object principal plane of a system, and the image principal plane are the two planes that:

- Have a point to point imaging from the first to the second.
- The magnification G is 1.
- Using some matrix calculations, the values are:

$$h_i = \frac{1 - M_{11}}{M_{21}} \tag{7.5}$$

$$h_o = \frac{\det \mathbf{M} - M_{22}}{M_{21}} \tag{7.6}$$

- h_o is positive when upstreams, and h_i when downstream, but those values could be negatives.

Achromatic System

- A system where the condition $M_{16} = M_{26} = 0$ is fulfilled is called achromatic. In this case, $\mathbf{M}$ is the full 6x6 matrix of the system, including both planes and dispersion component.
- Is a system that does not create dispersion.
- Important system in the design of lattices.

Equation of motion

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Reminder of basic equations of motion

In previous lectures, you have learned that in the moving coordinate system, the Lorentz equation

$$\frac{d}{dt}\vec{v} = \frac{e}{m}\left(\vec{v} \times \vec{B}\right) \tag{8.1}$$

becomes the following two uncoupled equations:

$$\frac{d^2x}{ds^2} - \left(k(s) - \frac{1}{\rho^2}\right)x = \frac{1}{\rho}\frac{\Delta p}{p_0} \tag{8.2}$$

$$\frac{d^2y}{ds^2} + k(s)y = 0 \tag{8.3}$$

where:

$$k(s) = \frac{1}{B\rho}\frac{\partial B_y(s)}{\partial x}$$

ρ is the radius of curvature of the electrons

$\frac{\Delta p}{p_0}$ is the momentum deviation respect the reference particle

If we concentrate in the on-energy particle ($p = p_0$), both equations 8.2 and 8.3, can be written as:

$$u'' + K(s)u = 0 \tag{8.4}$$

where u stands for x or y and $K(s)$ is given by:

$$K(s) = \begin{cases} -\left(k(s) - \frac{1}{\rho^2}\right) & u = x \\ k(s) & u = y \end{cases} \tag{8.5}$$

From here we can see that is difficult to focus simultaneously in both planes

Hill's equation

Equation 8.4 is the equation of an anharmonic oscillator (Hill's equation). To solve it, we can write it as:

$$\vec{u}' + \begin{pmatrix} 0 & -1 \\ K(s) & 0 \end{pmatrix} \vec{u} = 0 \tag{8.6}$$

where

$$\vec{u} = \begin{pmatrix} u \\ u' \end{pmatrix}$$

If K is constant, the solution can be writen as:

$$\vec{u}(s) = A\vec{u}_1(s) + B\vec{u}_2(s) \tag{8.7}$$

with

$$\vec{u}_1(s) = \begin{pmatrix} \sin(\sqrt{K}s) \\ \sqrt{K}\cos(\sqrt{K}s) \end{pmatrix} \quad \vec{u}_2(s) = \begin{pmatrix} \cos(\sqrt{K}s) \\ -\sqrt{K}\sin(\sqrt{K}s) \end{pmatrix} \tag{8.8}$$

and initial conditions:

$$\vec{u}_1(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \vec{u}_2(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{8.9}$$

and the transport map M is given by:

$$\vec{y}(s) = M(s - s_0) \times \vec{y}(s_0) \tag{8.10}$$

$$M(s - s_0) = \begin{pmatrix} \cos(\sqrt{K}(s - s_0)) & \sin(\sqrt{K}(s - s_0)) \\ -\sqrt{K}\sin(\sqrt{K}(s - s_0)) & \sqrt{K}\cos(\sqrt{K}(s - s_0)) \end{pmatrix}$$

is a rotation in the phase space.

K not constant

In case K is not constant (as in an accelerator), the Floquet theorem allows us to write the solution also as the linear combination of a "sinelike" and a "cosinelike" solutions, now wiht a variable amplitude and phase advance:

$$u(s) = Au_a(s) + Bu_b(s) \tag{8.11}$$

where

$$u_a(s) = \sqrt{C}\sqrt{\beta(s)}e^{+i\phi(s)} \tag{8.12}$$

$$u_b(s) = \sqrt{C}\sqrt{\beta(s)}e^{-i\phi(s)} \tag{8.13}$$

where C is a constant and $\beta(s)$ is the s-dependent amplitude, know as optical betatron function, with units of lenght (usually meters). The phase term $\phi(s)$ depends on $\beta(s)$ as:

$$\phi(s) = \int_{s_0}^{s} \frac{1}{\beta(\tau)} d\tau + \phi_0 \tag{8.14}$$

Substituing on of the equations 8.12 or 8.13 in equation 8.4, the following diferential equation for the beta function is obtained

$$\frac{1}{2}\beta(s)\beta''(s) - \frac{1}{4}\beta'^2(s) + K(s)\beta^2(s) = 1 \tag{8.15}$$

Periodic System

- Storage rings are, by definition, periodic:

$$\vec{B}(s) = \vec{B}(s + C) \tag{9.1}$$

  where C is the circumference of the machine.
- In this case, the description of the motion of the particles using the betatron functions is much more practical than the one using trajectories. Why?
- The trajectories are not periodic. The optical functions are periodic:

$$\beta_{x,y}(s) = \beta_{x,y}(s + C) \tag{9.2}$$

- Also, we have now a periodic dispersion η.
- And a periodic transfer matrix.

Betatron function

For a periodic system, as in the case of a storage ring $K(s) = K(s+L)$ and the beta function is also periodic, $\beta(s) = \beta(s+L)$, and the total phase advance per revolution is the tune or number of oscillations:

Definition

Tune

$$Q = \frac{1}{2\pi}\oint \frac{1}{\beta(\tau)}d\tau$$

Going back to the equations of motion, we can combine equations 8.12 and 8.13 in the two independent solutions, sinelike and cosinelike:

$$S(s) = -\frac{i}{2}\left(u_a(s) - u_b(s)\right) \quad (9.3)$$

$$C(s) = \frac{1}{2}\left(u_a(s) + u_b(s)\right) \quad (9.4)$$

with the same initial conditions.

Definition

Twiss functions

beta function	$\beta(s)$
alpha function	$\alpha(s) \equiv -\frac{1}{2}\frac{d\beta(s)}{ds}$
gamma function	$\gamma(s) \equiv \frac{1+\alpha^2(s)}{\beta(s)}$

The sinelike and cosinelike solutions are:

$$\vec{S}(s) = \begin{pmatrix} \sqrt{\beta(s)\beta(s_0)}\sin(\phi(s)+\phi_0) \\ \frac{\sqrt{\beta(s_0)}}{\sqrt{\beta(s)}}\left(\cos(\phi(s)+\phi_0)+\alpha(s)\sin(\phi(s)+\phi_o)\right) \end{pmatrix} \tag{9.5}$$

$$\vec{C}(s) = \begin{pmatrix} \frac{\sqrt{\beta(s)}}{\sqrt{\beta(s_0)}}\left(\cos(\phi(s)+\phi_0)+\alpha(s_0)\sin(\phi(s)+\phi(s_0))\right) \\ -\frac{1+\alpha(s)\alpha(s_0)}{\sqrt{\beta(s)\beta(s_0)}}\left(\sin(\phi(s)-\phi_0)+(\alpha_0-\alpha)\cos(\phi(s)+\phi_0)\right) \end{pmatrix} \tag{9.6}$$

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One turn transfer matrix $\mathcal{M}$

- The one turn transfer matrix $\mathcal{M}$ is given by:

$$\mathcal{M}_s = \mathbf{M}(s \mapsto (s+C)) \tag{9.7}$$

$$\vec{u}(s+C) = \mathcal{M}_s \times \vec{u}(s_0) \tag{9.8}$$

$$\mathcal{M}_s = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix} \tag{9.9}$$

- And it can be written as:

$$\mathcal{M} = \mathbf{I}\cos(2\pi Q) + \mathbf{J}\sin(2\pi Q) \tag{9.10}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{J}$ is

$$\mathbf{J} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \tag{9.11}$$

Combining the equation of motion and the definiton of the twiss parameters, we can obtain the followin equation for the invariant of the motion:

$$C = \gamma u^2 + 2\alpha u u' + \beta u'^2 \tag{9.12}$$

that is the equation of an ellipse in the phase space, where the area is:

$$\text{Area} = \pi C = \pi\varepsilon \tag{9.13}$$

Definition

ε is the emittance

At any given point, the rms beam size and divergence can be written as:

$$\sigma(s) = \sqrt{\varepsilon\beta(s)} \tag{9.14}$$

$$\sigma'(s) = \sqrt{\varepsilon\gamma(s)} \tag{9.15}$$

Periodic Dispersion

- Remember, the dispersion D is the trajectory that the off-energy particle with $\delta_p = 1$ and initial conditions $\vec{x} = (0; 0)$.
- This is the term that is present in the 3×3 transport matrix.
- Now we have the periodic dispersion η, that satisfies the equation:

$$\begin{pmatrix} \eta(s) \\ \eta'(s) \\ 1 \end{pmatrix} = \mathcal{M}_s \begin{pmatrix} \eta(s) \\ \eta'(s) \\ 1 \end{pmatrix} \tag{9.16}$$

$$= \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} \eta(s) \\ \eta'(s) \\ 1 \end{pmatrix} \tag{9.17}$$

Thanks for the attention

Thanks for your attention

For more information, and a copy of the uptodate presentation, check the web page:
`http://www.cells.es/Divisions/Accelerators/courses-and-presentations/master/`

Exercises and problems

Exercises I

1. The dipole magnets for the ALBA machine have a length of 1.4 m. The energy of the electrons stored on it is of 3 GeV. The number of dipoles is 32. What is the bending radius? What is the dipolar field?
2. A booster synchrotron is used to accelerate electrons between the Linac and the main storage ring. Let's assume a booster with 24 dipoles of 1 meter, where the field varies between 0.0417 T and 1 T. What is the variation of bending radius? and in the energy?
3. - Consider a system composed of a thin lens QD of focal length f_1 (defocusing), drift space L of length l and another thin lens QF of focal length f_2 (focusing):

   QD L QF

   what is the total transfer matrix for the system? What is the focal length?

Exercises II

- Using the last matrix, setting the two lenses to the same strength (still one focusing and one defocusing). Is the system focusing?
- If an object is located at a distance p upstream, where is the image?

4. **FODO CELL**: Consider a defocusing QD quadrupole sandwiched between two focusing quads QFh. The focal length of this one is half of the other. The system is:

$$\mathcal{M}_{FODO} = \mathbf{M}_{QFh} \times \mathbf{M}_L \times \mathbf{M}_{QD} \times \mathbf{M}_L \times \mathbf{M}_{QFh}$$

   - Write the individual transfer matrices.
   - Write the total transfer matrix.
   - Write the focal length.
5. How can you transport the one turn transfer matrix around a ring?
6. How can you transport the periodic dispersion around a ring?